

Lecture 10

In lecture 9 we derived RG flow equations for φ^4 model to $O(\epsilon)$:

$$\left\{ \begin{array}{l} \frac{dr}{dl} = 2r + \frac{\eta}{2} \frac{S_d}{(\bar{m})^d} \frac{\Lambda^d}{r+\Lambda^2} \\ \frac{du}{dl} = (\eta-d)u - \frac{3u^2}{2} \frac{S_d}{(\bar{m})^d} \frac{\Lambda^d}{(r+\Lambda^2)^2} \end{array} \right.$$

Find the nongaussian fixed point explicitly.

$$\frac{dr}{dl} = \frac{du}{dl} = 0$$

From the second equation we have:

$$U^* \approx \frac{2}{3}(\eta-d) \frac{(\bar{m})^d}{S_d} \Lambda^{4-d} \approx \frac{2}{3}\epsilon \frac{(\bar{m})^4}{S_4}$$

$$S_4 = \frac{\bar{m}^{4/\eta}}{\Gamma(\frac{4}{\eta})} = \frac{\bar{m}^2}{\Gamma(2)} = \bar{m}^2$$

$$U^* = \frac{16\pi^2\epsilon}{3}$$

$$r^* = -\frac{U^*}{4} \frac{S_4}{(\bar{m})^4} \Lambda^2 = -\frac{\epsilon}{6} \Lambda^2 \quad - \text{Wilson-Fisher fixed point.}$$

Note that $r^* < 0$ at transition - effect of fluctuation.

Note the following important property of the RF flow equation.

Recall that dimensions of r and u in units of
The wave length are given by:

$$\dim[r] = 2$$

$$\dim[u] = 4 - d = \varepsilon$$

Now the coefficients of the ~~dimensions~~ terms in the RHS of the flow equation are equal to the scalar dimensions of the corresponding quantities.

Let's linearize the flow equations around the fixed points.

First linearize around the gaussian fixed point.

$$r^* = u^* = 0$$

$$\text{Define } \delta r = r - r^*, \quad \delta u = u - u^*$$

$$\text{let } K_d = \frac{S_d \Lambda^d}{(2\pi)^d} \quad \text{for brevity.}$$

Linearized flow equations are given by,

$$\frac{d\delta r}{dl} = 2\delta r + \frac{\delta u}{2} K_d \Lambda^{-2}$$

$$\frac{d\delta u}{dl} = \varepsilon \delta u$$

$$\frac{d}{dt} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} K_d \lambda^{-2} \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

$\underbrace{\quad\quad\quad}_{M}$

The eigenvalues of M are:

$$\lambda_+ = 2, \lambda_- = \varepsilon$$

Corresponding eigenvectors:

$$\frac{1}{2} K_d \lambda^{-2} v_{t_2} \quad \text{(circled and crossed out)} = 0 \Rightarrow v_{t_2} = 0$$

$$\text{Thus } \vec{v}_t = (1, 0)$$

$$(2 - \varepsilon) v_{u_1} + \frac{1}{2} K_d \lambda^{-2} v_{u_2} = 0$$

$$\frac{v_{u_1}}{v_{u_2}} = - \frac{K_d \lambda^{-2}}{2(2 - \varepsilon)}$$

$$K_d = K_d = \frac{s_u}{(2m)^n} \lambda^n = \frac{\lambda^4}{8\pi^2}$$

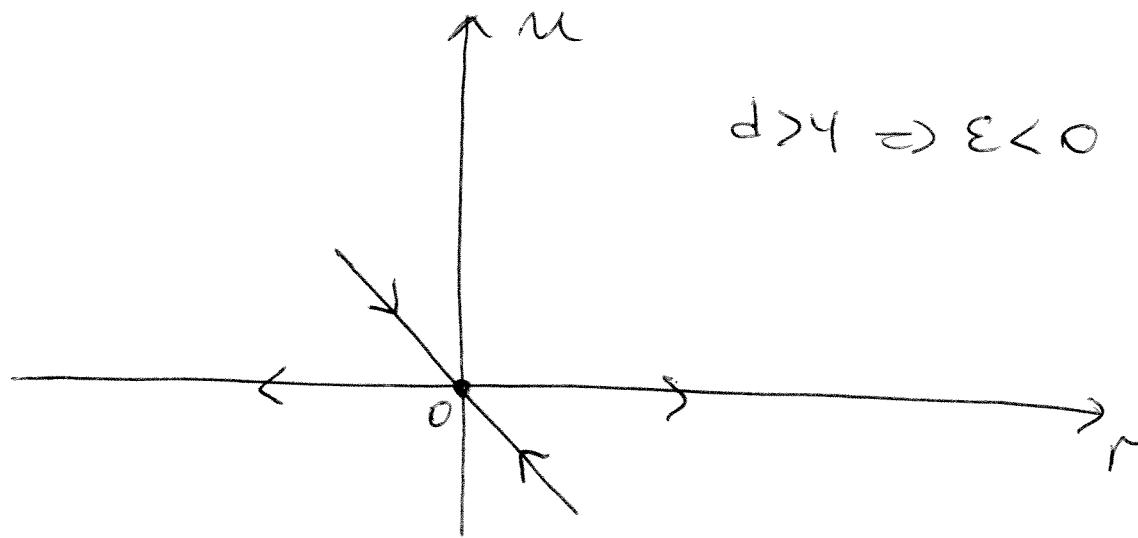
$$\vec{v}_2 = \left(-\frac{K_d \lambda^{-2}}{2(2 - \varepsilon)}, 1 \right) \approx \left(-\frac{\lambda^2}{32\pi^2}, 1 \right)$$

$$(\delta r, \delta u) = t \vec{v}_t + \tilde{u} \vec{v}_{\tilde{u}}$$

$$\left\{ \begin{array}{l} \frac{dt}{dl} = 2t \\ \frac{d\tilde{u}}{dl} = \epsilon \tilde{u} \end{array} \right.$$

$$t(l) = e^{2l} t(0)$$

$$\tilde{u}(l) = e^{\epsilon l} \tilde{u}(0)$$



For $d>4$ gaussian fixed point is stable.

What it means is that there is only one unstable direction $t=r$ - This is always unstable since changing r means changing temperature - any change in temperature, keeping everything else fixed leads away from the central point.

The variables for which small deviations from fixed point grow under renormalization are called relevant. r or t is a relevant variable.

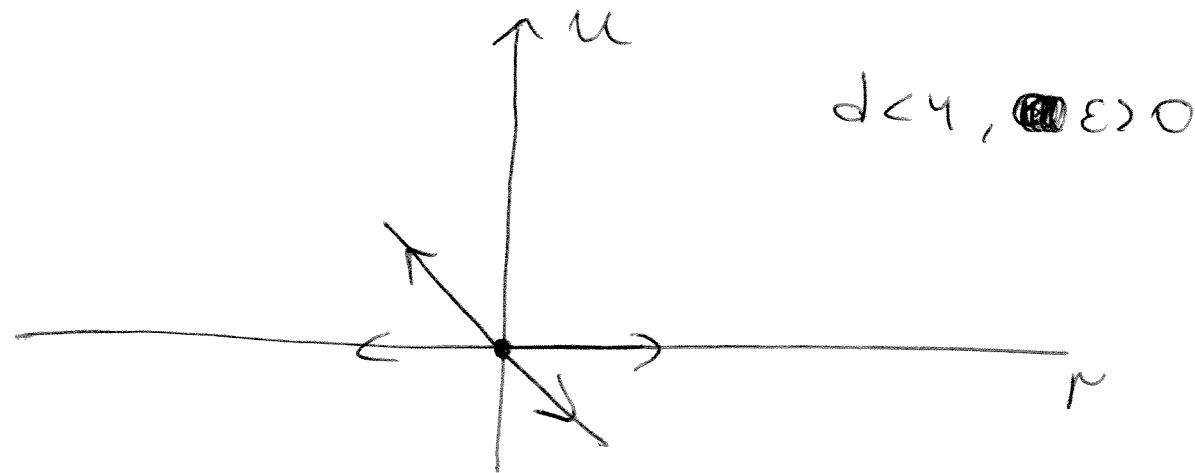
Relevant variables are associated with positive eigenvalues

Direction \vec{v} for $d > 4$ is stable - small deviations are suppressed exponentially.

Such variables are called irrelevant.

~~Wilson-Fisher fixed point~~

Wilson-Fisher fixed point is unphysical in $d > 4$ since $u^* < 0$.



When $d < 4$ both directions have positive eigenvalues \Rightarrow

\Rightarrow gaussian fixed point is unstable \Rightarrow any deviation from it leads away from the fixed point under RG transformation.

Let's now analyze the stability of the Wilson-Fisher fixed point

Introduce beta-function:

$$\frac{dr}{dl} = \beta_r(r, u)$$

$$\frac{du}{dl} = \beta_u(r, u)$$

$$\vec{\beta}(r, u) = (\beta_r, \beta_u) \text{ - Beta-funktion.}$$

In our case:

$$\beta_r(r, u) = 2r + \frac{\gamma}{2} \frac{K_d}{r + \Lambda^2}$$

$$\beta_u(r, u) = \varepsilon u - \frac{3u^2}{2} \frac{K_d}{(r + \Lambda^2)^2}$$

$$r^* = -\frac{\varepsilon}{6} \Lambda^2, \quad u^* = \frac{16\pi^2}{3} \varepsilon$$

$$\delta r = r - r^*, \quad \delta u = u - u^*$$

$$\frac{d}{dl} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \beta_r}{\partial r} & \frac{\partial \beta_r}{\partial u} \\ \frac{\partial \beta_u}{\partial r} & \frac{\partial \beta_u}{\partial u} \end{pmatrix}_{r^*, u^*}}_M \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

$$\frac{\partial \beta_r}{\partial r} = 2 - \frac{u^*}{2} K_d \Lambda^{-4} = 2 - \frac{\varepsilon}{3}$$

$$\frac{\partial \beta_r}{\partial u} = \frac{1}{2} K_d \Lambda^{-2} = \frac{\Lambda^2}{16\pi^2}$$

$$\frac{\partial \beta_u}{\partial r} = 3u^{*2} K_d \Lambda^{-6} = O(\varepsilon^2) \text{ - neglected}$$

$$\frac{\partial \beta_n}{\partial u} = \varepsilon - 3u^* K_d \Lambda^{-4} = \varepsilon - 3 \frac{16\pi^2 \varepsilon}{3} \frac{1}{8\pi^2} = -\varepsilon$$

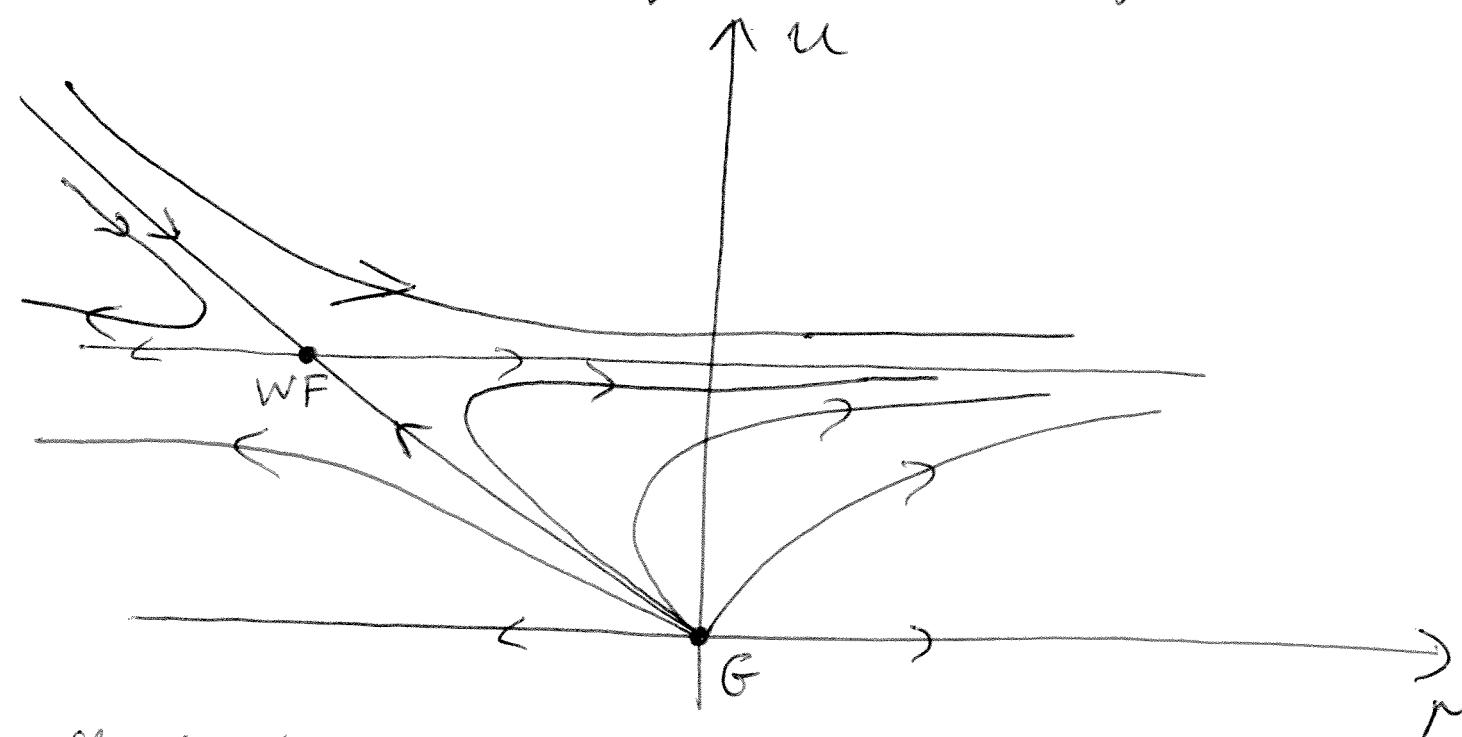
Thus we obtain:

$$M = \begin{pmatrix} 2 - \frac{\varepsilon}{3} & \frac{\Lambda^2}{16\pi^2} \\ 0 & -\varepsilon \end{pmatrix}$$

The eigenvalues are: $\lambda_+ = 2 - \frac{\varepsilon}{3}$, $\lambda_- = -\varepsilon$.

Eigenvectors: $\tilde{v}_+ = (1, 0)$

$\tilde{v}_- = \left(-\frac{\Lambda^2}{32\pi^2}, 1 \right)$ — same as for the gaussian fixed point.



All random functionals with (r, u) along the line connecting the G and the WF fixed points will flow toward the WF fixed point under RG.

Other points will either flow toward $\Gamma \rightarrow \infty$ - paramagnetic phase with $\beta = 0$, or $\Gamma \rightarrow -\infty$ - ferromagnetic phase with $\beta = 0$.

The line connecting G and wF fixed points is the basin of attraction of the wF fixed point. All random functions in the basin of attraction have the same critical properties universality.

Consider terms in the random functional that we have neglected - now we can justify this.

$$2. v \int d^d x \varphi^6$$

Calculate the dimension of v .

$$\dim[v] - d + 6 \frac{d-2}{2} = 0$$

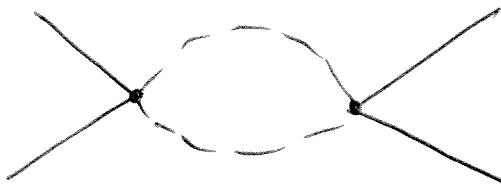
$$\dim[v] = -2d + 6 = 2\varepsilon - 2$$

Thus RG flow equation for v will have the form

$$\frac{dv}{dl} = (-2+2\varepsilon)v + O(u^2, v^2)$$

Since the ~~the~~ coefficient of the leading term is negative, it is clear that if we start from a random function with a small v , it will flow to 0 under RG. Thus v is an irrelevant variable near U dimensions.

2. We have also neglected momentum dependence of u , which comes from this term:



$$\left(\frac{u}{4!}\right)^2 \int_{k_1, \dots, k_4} (\text{volume})^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \varphi_c(\vec{k}_1) \dots \varphi_c(\vec{k}_4)$$

$$\cdot \int_{\Lambda/2}^{\Lambda} \frac{dq}{(\text{volume})^d} \frac{1}{r+q^2} \frac{1}{r+(\vec{q}+\vec{k}_1+\vec{k}_2)^2}$$

This gives us u , which is a function of \vec{k}_1 and \vec{k}_2 :

$u(\vec{k}_1, \vec{k}_2)$. Expand in Taylor series with respect to \vec{k}_1 and \vec{k}_2 .

$$u(\vec{k}_1, \vec{k}_2) \approx u_0 + \overset{w}{\underset{\vec{k}_1 \cdot \vec{k}_2}{\text{---}}} + \dots$$

In real space, the leading momentum-dependent terms will give rise to terms of the form:

$$w \int d^d x (\vec{\nabla} \varphi)^2 \varphi^2$$

$$\text{dim}[w] - d + 4 \frac{d-2}{2} + 2 = 0.$$

$$\text{dim}[w] = d - 2d + 4 - 2 = 2 - d = \epsilon - 2$$

Thus $\text{dim}[w] < 0$ and w is also irrelevant.

Now let's calculate the critical exponents.

Consider correlation function:

$$\text{F}(\vec{x}) = \langle \varphi(\vec{x}) \varphi(0) \rangle - \langle \varphi(\vec{x}) \rangle \langle \varphi(0) \rangle$$

At a given point in the (r, n) space F is a function of ~~r~~ r and n , or t and \tilde{n} :

$$\frac{dt}{dl} = d_t t$$

$$\frac{d\tilde{n}}{dl} = d_{\tilde{n}} \tilde{n}$$

~~Consider~~ Suppose we start from point (t, \tilde{n}) and move along the RG flow for "time" l .

$$t \rightarrow t e^{dt l}, \quad \tilde{n} \rightarrow \tilde{n} e^{d\tilde{n} l}, \quad \vec{x} \rightarrow \frac{\vec{x}}{l}, \quad b = l,$$

~~Consider~~

$$\varphi'(\vec{k}') = b^{-\frac{d+2-\gamma}{2}} \varphi_c(k^*)$$

$$\varphi'(\vec{x}') = \int \frac{d^d k'}{(2\pi)^d} \varphi'(\vec{k}') e^{i \vec{k}' \cdot \vec{x}'} =$$

$$= \int \frac{d^d k}{(2\pi)^d} b^d b^{-\frac{d+2-\gamma}{2}} \varphi_c(k^*) e^{i \vec{k} \cdot \vec{x}'} =$$

$$= b^{\frac{d-2+\gamma}{2}} \varphi_c(\vec{x})$$

$$\text{Ans } \varphi \rightarrow \varphi b^{\frac{d-2+\gamma}{2}}$$

10

thus we have:

$$G\left(\frac{x}{b}, t e^{d\ell}, \tilde{u} e^{d\ell}\right) = b^{d-2+\gamma} G(x, t, \tilde{u})$$

or:

$$G(x, t, \tilde{u}) = b^{-(d-2+\gamma)} G\left(\frac{x}{b}, t e^{d\ell}, \tilde{u} e^{d\ell}\right)$$

Take $t, \tilde{u} = 0$ - fixed point.

$$G(x, 0, 0) = b^{-(d-2+\gamma)} G\left(\frac{x}{b}, 0, 0\right)$$

It follows that:

$$f(x, 0, 0) \sim \frac{1}{x^{d-2+\gamma}}$$

This is why γ is called anomalous dimension.

~~Move away from the fixed point in the t -direction.~~

$$G(x, t, 0) = b^{-(d-2+\gamma)} G\left(\frac{x}{b}, t e^{d\ell}, 0\right)$$

~~choose $b \neq x$~~

Move away from the fixed point in the t -direction:

$$f(x, t, 0) = b^{-(d-2+\gamma)} f\left(\frac{x}{b}, t b^{d\ell}, 0\right)$$

This must hold for any b .

choose $b = x$.

$$G(x, t, 0) = x^{-(d-2+\gamma)} G(1, t x^{\lambda_t}, 0)$$

On the other hand, we know that:

$$G(x, t) = x^{-(d-2+\gamma)} f\left(\frac{x}{\xi}\right), \text{ where } \xi \text{ is one correlation length.}$$

$$\text{This means that } \xi \sim |t|^{-\lambda_t^{-1}}$$

Thus the correlation length critical exponent is given by:

$$\nu = \lambda_t^{-1} = \frac{1}{2 - \frac{\epsilon}{3}} \approx \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$$

In MFT $\nu = \frac{1}{2}$, exact value for 3D Ising model

$\nu = \cancel{0.63} \approx 0.63$, $O(\epsilon)$ result $\nu \approx 0.58$.

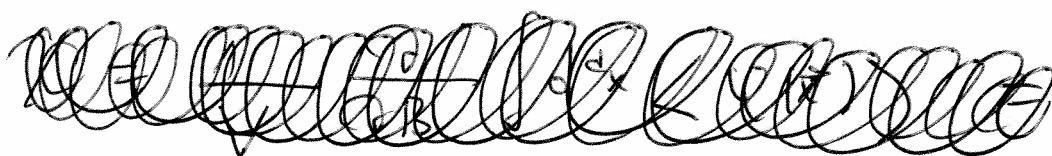
Calculate the susceptibility critical exponent γ .

$$S[\varphi] = \int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \varphi^2 + \frac{1}{4!} \varphi^4 - B \varphi \right]$$

Note that the correlation function may be evaluated as:

$$\begin{aligned} \mathcal{G}(\vec{x} - \vec{x}') &= \left. \frac{\delta \langle \varphi(\vec{x}) \rangle}{\delta B(\vec{x}')} \right|_{B=0} = \\ &= \frac{\delta}{\delta B(\vec{x}')} \frac{1}{Z} \int D\varphi \varphi(\vec{x}) e^{-S[\varphi]} = \\ &= \langle \varphi(\vec{x}) \varphi(\vec{x}') \rangle - \langle \varphi(\vec{x}) \rangle \langle \varphi(\vec{x}') \rangle \end{aligned}$$

On the other hand, the magnetic susceptibility is given by:



$$\begin{aligned} \bullet \quad \chi &= \frac{1}{V} \frac{\partial \langle \varphi \rangle}{\partial B} = \frac{1}{V} \int d^d x \mathcal{G}(\vec{x}) = \\ &\text{(Diagram of a magnetic dipole)} = \frac{1}{V} \int d^d x x^{-(d-2+y)} f\left(\frac{x}{\xi}\right) \end{aligned}$$

Let $y = \frac{x}{\xi}$. Then we obtain:

$$\chi = \frac{1}{V} \int d^d y \xi^d \xi^{-(d-2+y)} y^{-(d-2+y)} f(y)$$

$$\text{Thus } \chi \sim \xi^{2-y}$$

$$\xi \sim |t|^{-\nu}, \quad \chi \sim |t|^{-\gamma}$$

thus $\gamma = (2-\nu)\nu$ - scaling relation.

To 0(ε) we obtain:

$$\gamma = 2\nu = 1 + \frac{\varepsilon}{6}$$

At $\varepsilon=1$ $\gamma = 1.17$. Exact $\gamma \approx 1.24$

Find the order parameter critical exponent.

$$\langle \varphi \rangle \sim (-t)^\beta$$

$$\text{let } g(t, \tilde{u}) = \langle \varphi \rangle_{t, \tilde{u}}$$

By the same arguments as in the case of the correlation function ~~function~~, we obtain:

$$g(t, \tilde{u}) = \delta^{-\frac{d-2+\gamma}{2}} g(t \delta^{dt}, \tilde{u} \delta^{d\tilde{u}})$$

~~Fix~~ Fix $\tilde{u}=0$.

$$g(t, 0) = \delta^{-\frac{d-2+\gamma}{2}} g(t \delta^{dt}, 0)$$

~~choose~~

~~choose~~

$$\text{choose } \delta = t^{-\frac{1}{d\epsilon}}$$

$$g(t,0) = t^{\frac{d-2+\gamma}{2\lambda_+}} g(1,0)$$

$$\frac{1}{\lambda_+} = \nu$$

Thus $\beta = \frac{d-2+\gamma}{2} \nu$ - another scaling relation.

$$\text{To } O(\varepsilon) \quad \beta = \frac{d-2}{2} \nu = \frac{1}{2} - \frac{\varepsilon}{6}$$

At $\varepsilon=1$, $\beta \approx 0.33$. Exact $\beta \approx 0.328$