Homework 4
Due Wednesday, March 3

1. *Goldstone mode dispersion.*

   (a) Consider Heisenberg model on a 3-dimensional cubic lattice:

   $$H = -\frac{J}{2}\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$

   where i and j are nearest-neighbor sites and $\mathbf{S}_i$ are 3-component unit vectors. Recall that the equation of motion for a spin can generally be written as:

   $$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times \mathbf{B}_i,$$

   where $\mathbf{B}_i$ is the total magnetic field acting on the spin at site i. The field in this case is due to the nearest-neighbor spins, interacting with a given spin, and can be calculated as:

   $$\mathbf{B}_i = -\frac{\partial H}{\partial \mathbf{S}_i} = J\sum_j \mathbf{S}_j,$$

   where the sum over j is restricted to the nearest neighbors of i. Assume that $T = 0$ and the spins are polarized in the z-direction. Using the equation of motion above find the dispersion $\omega(\mathbf{k})$ of small fluctuations (Goldstone modes) around the fully polarized state. *Hint: linearize the equations of motion with respect to fluctuations and find the eigenmodes.*

   (b) Repeat the same calculation for the XY-model on the cubic lattice. Compare the Goldstone mode dispersions of the Heisenberg and XY models for small momenta $\mathbf{k}$.

2. *XY-model with anisotropy.* Consider the following XY-model on the 2-dimensional square lattice:

   $$H = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) + A\sum_i \sin^2(\theta_i).$$

   The second term introduces anisotropy with respect to spin $[S_i^x = \cos(\theta_i), S_i^y = \sin(\theta_i)]$ rotations, and at zero temperature there are only two degenerate ground states: $\theta = 0$ and $\theta = \pi$.

(a) Consider a state with two domains: $\theta(x, y) \to \pi$ when $x \to -\infty$ and $\theta(x, y) \to 0$ when $x \to \infty$. Find the domain wall profile, i.e. $\theta(x)$. *Hint: assume θ varies slowly on the scale of the lattice spacing and write down the continuum form of H. Assuming that θ depends only on x, minimize the continuum energy functional variationally. Solve the resulting differential equation for θ with the boundary conditions $\theta(-\infty) = \pi, \theta(\infty) = 0$.* What is the domain wall width as a function of J/A?

(b) Find the energy of the domain wall per unit length.

(c) Now instead of a state with two domains, consider a state with a single vortex with a unit winding number. Argue that the spin distribution far away from the vortex core can be described as two domains, each with (approximately) uniform $\theta = \pi, 0$. The domains are separated by a domain wall, running through the vortex core. Phase gradient is mainly concentrated in the domain wall. Estimate vortex energy as a function of the system size.

(d) Now consider a vortex-antivortex pair a large distance R apart. Sketch the spin distribution in the plane. Estimate the energy of the pair as a function of R.

(e) Using the above results, discuss the possibility of a vortex-antivortex unbinding transition in this system at finite temperature.