Sheaf cohomology*

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Derived functor cheat sheet

If $\mathcal{A}$ is a "nice" abelian category, $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, then there exists a sequence

$$R^i F, \quad i \geq 0$$

of functors such that:

- $R^0 F \cong F$
- If $0 \to A \to B \to C \to 0$ is exact, then there is a long exact sequence

$$0 \to \underbrace{F(A)}_{=R^0 F(A)} \to F(B) \to F(C) \to R^1 F(A) \to R^1 F(B) \to R^1 F(C) \to \dots$$

- If F is exact, then $R^i F = 0$ for $i \geq 1$.

Namely, if $A \in \mathrm{Ob}(\mathcal{A})$, pick an injective resolution

$$0 \to A \to I^0 \to I^1 \to \dots$$

and define $R^i F$ by $R^i F(A) = H^i(F(I^\bullet))$. Note that if I is injective then $R^i F(I) = 0$ for all $i > 0$. Also, these are well-defined additive functors for all i.

Here, having enough injectives is good enough to be "nice" (however "nice" is actually weaker).

If we fix some nice category of sheaves on X i.e. $\mathfrak{Ab}(X)$, R-Mod(X), $\mathcal{O}(X)$-Mod etc, then:

Proposition. These all have enough injectives.

*Synthesized from Erik's notes and the notes I (mlbaker) took.

Sheaf cohomology

Definition. The functor of "global sections"

$$F \mapsto F(X) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$$

will be denoted $\Gamma(X, -)$.

Definition (attempt)**.** The ith cohomology functor

$$H^i : \mathfrak{Ab}(X) \to \mathfrak{Ab}$$

is $R^i\Gamma(X, -)$.

Proposition. $\Gamma(X, -)$ is left exact.

Proof. Suppose

$$0 \to F \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 0$$

is exact. We need to show that

$$0 \to F(X) \xrightarrow{\varphi_X} G(X) \xrightarrow{\psi_X} H(X)$$

is exact. Note that

$$\ker(\varphi_X) = (\ker \varphi)_X = (0)_X = 0.$$

Also,

$$\operatorname{im}(\varphi_X) \subseteq \ker(\psi_X) \quad \text{since } \psi \circ \varphi = 0$$

so that $\psi_X \circ \varphi_X = 0$. Suppose $s \in \ker(\psi_X) = (\ker \psi)_X = (\operatorname{im} \varphi)_X$. The rest of the proof works out by arguing that things can be glued together: recall

$$\operatorname{im} \varphi := (\operatorname{im}_{\text{psh}} \varphi)^+ \quad \text{(where + denotes sheafification)}$$

so s is locally a section of $\operatorname{im}_{\text{psh}} \varphi$, so it's locally the image of a section of F. Since sections of φ are injective, we get uniqueness, so we can glue to get a section of F. □

Remark. This fails to prove right exactness since we need injectivity to lift sections of epic sheaf maps, otherwise local sections might not agree on overlaps.

The global sections functor for presheaves turns out to be exact, so you can't get an interesting cohomology theory in that setting.

Cohomology is all about measuring "failures of right-exactness", i.e. "obstructions to lifting global sections"; since local lifts are always fine, this basically amounts to "obstructions to gluing local constructions together".

Example. If $X \subseteq \mathbb{C}$ is connected and open (or X is a Riemann surface), we have a short exact sequence

$$0 \to 2\pi i\mathbb{Z} \xrightarrow{\text{inclusion}} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1$$

where $2\pi i\mathbb{Z}$ is the constant sheaf $\mathbb{Z}$, $\mathcal{O}$ is the sheaf of holomorphic functions, and $\mathcal{O}^*$ is the sheaf of nonzero holomorphic functions. Also, 1 is the same sheaf as 0, except written multiplicatively.

If we have a nonzero holomorphic function on some open set, then given any point in that set, we can obtain a logarithm for the function on some sufficiently small neighbourhood.

Now consider global sections:

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}(X) \xrightarrow{\exp} \mathcal{O}^*(X) \to 1$$

where the map $\mathcal{O}(X) \to \mathcal{O}^*(X)$ is not necessarily surjective (on $\mathbb{C} \setminus \{0\}$, the map $(z \mapsto z) \in \mathcal{O}^*(X)$ has no global lift), so the sequence is not in general exact.

We saw before that if $0 \to F \to G \to H \to 0$ is exact, we get

$$0 \to \Gamma(X, F) \to \Gamma(X, G) \to \Gamma(X, H) \to H^1(X, F) \to H^1(X, G) \to H^1(X, H) \to \dots$$

This is *the* fundamental property of sheaf cohomology from an algebraic point of view. If G is *acyclic* then this characterizes $H^i(X, H)$.

If we replace the 1 above with the first cohomology groups, then we get the following exact sequence:

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}(X) \xrightarrow{\exp} \mathcal{O}^*(X) \to H^1(X, 2\pi i\mathbb{Z}) \to H^1(X, \mathcal{O}) \to \dots$$

with

$$H^1(X, 2\pi i\mathbb{Z}) = \frac{\mathcal{O}^*(X)}{\operatorname{im}(\exp)_X}, \qquad \underbrace{H^1(X, \mathcal{O}) = 0}_{\text{if } X=\mathbb{C}\setminus\{x_1,\dots,x_n\}}.$$

Now set $X = \mathbb{C} \setminus \{x_1, \dots, x_n\}$. We ask the question: what is $\operatorname{im}(\exp_X)$? Just use complex analysis. Locally, we can always find a logarithm for a nonvanishing holomorphic function: suppose $f \in \mathcal{O}^*(X)$. Then $\log f$ exists locally, and

$$\frac{d}{dz} \log f = \frac{f'(z)}{f(z)}$$

so locally

$$\log f = \int_{z_0}^{z} \frac{f'(w)\, dw}{f(w)}.$$

When is this integral single-valued? Well, this will happen if and only if each residue of f/f' is 0. Near z_0, write

$$f(z) = (z - z_0)^n \cdot g(z), \quad g(z) \neq 0$$

so that

$$f'(z) = n(z - z_0)^{n-1} \cdot g(z) + (z - z_0)^n \cdot g'(z)$$

hence

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic in a nhd of } z_0}$$

and therefore we have that the residue is simply

$$\operatorname{Res}_{z_0}\left(\frac{f'}{f}\right) = n.$$

This is 0 if $z_0 \in X$. It could only be nonzero if z_0 is one of our "removed points" x_i. So $f \in \operatorname{im}\exp_X$ if and only if $\operatorname{Res}_{x_i}(f'/f) = 0$.

Note that $f \mapsto \operatorname{Res}_{x_i}(f'/f) : \mathcal{O}^*(X) \to \mathbb{Z}$ is a group homomorphism. Assume $x_i = 0$ to make the calculation easier; then we have

$$\frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

This is exactly the same situation that happens in algebraic curves: if you have a smooth curve at a point, then the local ring there is a DVR (discrete valuation ring).

$$\varphi : f \mapsto \left(\operatorname{Res}_{x_i}\frac{f'}{f}\right)_{i=1}^{n} : \mathcal{O}^*(X) \to \mathbb{Z}^n$$

so $\operatorname{im}\exp = \ker\varphi$. So

$$H^1(X, 2\pi i\mathbb{Z}) = \frac{\mathcal{O}^*(X)}{\operatorname{im}\exp} = \frac{\mathcal{O}^*(X)}{\ker\varphi} = \mathbb{Z}^n$$

and here n is the number of holes we've punctured. We might hope sheaf cohomology is some kind of generalisation of "hole counting" (which we already saw how to do last time with algebraic topology).

Example (de Rham cohomology)**.** Let X be a smooth manifold, Ω^k the sheaf of k-forms, then

$$0 \to \mathbb{R} \to C^\infty = \Omega^0 \xrightarrow{d} \Omega^1 \to \dots$$

is exact (by Poincaré) and forms an "acyclic resolution" for the constant sheaf $\mathbb{R}$. Thus

$$\dots \to \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \to \dots$$

and

$$H^k(X, \mathbb{R}) = \frac{\ker d_k}{\operatorname{im} d_{k-1}} = H^k_{\text{de Rham}}(X)$$

Remark. This is essentially a lifting obstruction problem.

Theorem. If X is "nice" (e.g. a manifold), then

$$H^i(X, \mathbb{Z}) \cong H^i_{\text{Čech}}(X),$$

and generally,

$$H^i(X, A) \cong H^i_{\text{Čech}}(X) \otimes A.$$

Proof. We'll see something more general later. □

Čech it again

Take an open cover; assume it's locally finite. Construct the nerve $N(\mathcal{U}) \in \Delta\mathbf{Set}$ by putting in a 0-simplex for $U \in \mathcal{U}$, a 1-simplex for $U \cap V \neq \varnothing$ ($U \neq V$), and in general, a k-simplex for every $U_0 \cap \dots \cap U_k \neq \varnothing$ with the U_i distinct in $\mathcal{U}$ $(*)$.

From $N(\mathcal{U})$, construct a cochain complex

$$\ldots \to C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \to \ldots$$

and

$$C^k \cong \bigoplus_{\substack{i_0 < \ldots < i_k \\ (*) \text{ holds}}} \underbrace{\mathbb{Z}(U_{i_0} \cap \ldots \cap U_{i_k})}_{\substack{\text{sections of const. sheaf } \mathbb{Z} \\ \text{over } U_{i_0} \cap \ldots \cap U_{i_k}}} .$$

To go from $C^{k-1} \to C^k$ we need to get maps

$$\mathbb{Z}(U_{i_0} \cap \ldots \cap \widehat{U}_{i_s} \cap \ldots \cap U_{i_k}) \to \mathbb{Z}(U_{i_0} \cap \ldots \cap U_{i_k})$$

where the hat denotes omission. Just take these maps to be multiplication by $(-1)^s$. If we assume all the intersections are connected then we simply have

$$\mathbb{Z}(U_{i_0} \cap \ldots \cap U_{i_k}) \cong \mathbb{Z}$$

and

$$\mathbb{Z}(U_{i_0} \cap \ldots \cap \widehat{U}_{i_s} \cap \ldots \cap U_{i_k}) \to \mathbb{Z}(U_{i_0} \cap \ldots \cap U_{i_k})$$

is just the sheaf restriction map.

This motivates a definition.

Definition. If $\mathcal{U}$ is a finite open cover of X, F is an abelian sheaf on X, define

$$C^k(\mathcal{U}, F) = \bigoplus_{\substack{i_0 < \ldots < i_k \\ U_{i_0} \cap \ldots \cap U_{i_k} \neq \varnothing}} F(U_{i_0} \cap \ldots \cap U_{i_k})$$

and also define $C^{k-1} \xrightarrow{d} C^k$ by

$$(-1)^s \cdot \mathrm{res}^{U_{i_0} \cap \ldots \cap \widehat{U}_{i_s} \cap \ldots \cap U_{i_k}}_{U_{i_0} \cap \ldots \cap U_{i_k}}(F) : F(U_{i_0} \cap \ldots \cap \widehat{U}_{i_s} \cap \ldots \cap U_{i_k}) \to F(U_{i_0} \cap \ldots U_{i_k})$$

glue these together to get $C^{k-1} \to C^k$ as desired.

If you hate yourself, check that $d^2 = 0$ so we indeed obtain a cochain complex.

Now define

$$H^k_{\check{\text{C}}\text{ech}}(\mathcal{U}, F) = \frac{\ker d^k}{\operatorname{im} d^{k-1}}$$

and then finally, as before, we defined the Čech cohomology of X by taking a colimit over all such open covers $\mathcal{U}$:

$$H^k_{\check{\text{C}}\text{ech}}(X, F) = \operatorname*{colim}_{\mathcal{U}} H^k_{\check{\text{C}}\text{ech}}(\mathcal{U}, F)$$

When X is "nice", these things agree with the derived functor cohomology. However we can actually *compute them*!

Directions from here

- Do the calculation for affine schemes and projective schemes.
- *Serre's criterion for affineness.*