2013.07.30

Cohomology theory: a thing that goes from a category and spits out a sequence of abelian groups.

We want to look at top. spaces you can get from gluing together triangles.

<u>Defn</u>: the standard n-simplex is

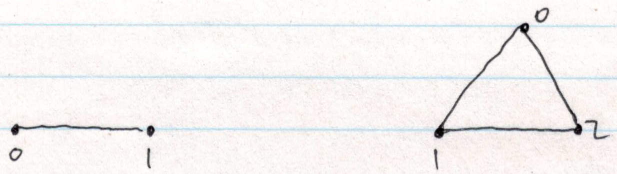
$$\Delta_n = \{ x \in \mathbb{R}^{n+1} \mid x_i \geq 0, \quad \sum_{i=0}^{n} x_i = 1 \}$$

where the standard basis of $\mathbb{R}^{n+1}$ is taken to be $\{e_0, \dots, e_n\}$.

If $f : \{0, \dots, n\} \longrightarrow \{0, \dots, m\}$, then define $\tilde{f} : \Delta_n \longrightarrow \Delta_m$ by

$$\tilde{f}\left(\sum_i a_i e_i\right) = \sum_i a_i f(e_i).$$

We'll put a restriction on these maps to make things nicer: we require them to be strictly monotone. This corresponds to allowing the maps to not degenerate things and preserve orientation, in a sense.



So we have a bunch of top. spaces and maps between them. These form a nice subcategory of the category of topological spaces.

<u>Defn</u>: Δ is the category with objects $\underset{[n]}{\{0, \dots, n\}}$ for $n \in \mathbb{N}$, and strict order-preserving maps as morphisms.

(One object for each isomorphism class of finite nonempty totally ordered set i.e. we pass to the "skeleton").

With this definition, what we'll end up with is the category of "delta-complexes" (see Hatcher). If we omit strictness we would get "simplicial sets".

<u>Defn</u>: the functor $\mathrm{Real}_0 : \Delta \to \mathrm{Top}$ is given by

$$[n] \longmapsto \Delta_n,$$
$$f \longmapsto \tilde{f}.$$

"gluing" in topology means you take a disjoint union of some spaces and quotient out. This is a <u>colimit</u>, So what we want to do is sort of "add colimits" to our category.

If A is a category, then

$$\mathrm{Psh}(A) := [A^{op}, \mathrm{Set}]$$

is the category of set-valued presheaves on A. We have the Yoneda embedding $Y : A \to \mathrm{Psh}(A)$

$$a \longmapsto \hom(-, a).$$

We now make (and refuse to prove) the statement that passing from a category to its category of set-valued presheaves is the most general way to adjoin colimits.

<u>Theorem</u>: If C is a category, then $\mathrm{Psh}(A)$ is cocomplete, $Y(C) \subseteq \mathrm{Psh}(A)$ is <u>dense</u> (in the sense that any set-valued contravariant functor can be written as a colimit of representable ones), and if $F : A \to B$ is some covariant functor, then there is an
(cocomplete)

$$\tilde{F} : \mathrm{Psh}(A) \longrightarrow B \quad \text{(unique up to unique natural isomorphism)}$$

such that

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "Y"'] & B \\ \mathrm{Psh}(A) \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

and $\tilde{F}$ is <u>cocontinuous</u> (i.e. preserves colimits).

Remember we wanted to take colimits of collections of triangles. So the thing to do is look at the presheaf category of Δ.

<u>Defn</u>: $Psh(\Delta)$ is called the category of <u>Δ-sets</u>. We will denote it by ΔSet.

Because of our theorem, the functor $Real_0$ extends to a cocontinuous functor defined on ΔSet:

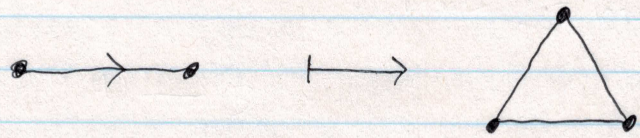
$$Real := \widetilde{Real_0}.$$

This part is just supposed to convince you that category theory is cool and will always give you the correct definitions if you wield it properly.

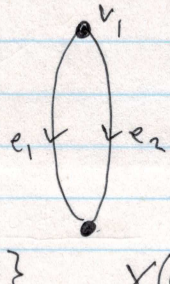
If $X \in \Delta Set$, we get
$X(n)$ for $n \geq 0$ ($X(n)$ = "collection of n-simplices in X")
and if $f : [n] \longrightarrow [m]$ is order-preserving then we get
$X(f) : X(m) \longrightarrow X(n)$ ("gluing maps")

Define $\Delta(n) = hom(-, [n])$ to be the <u>combinatorial standard n-simplex</u>.

Note
$$\Delta(n)(n) = hom([n], [n]) \text{ is a set of size } 1$$
$$\Delta(n)(0) = hom(\{0\}, [n])$$
$$\Delta(n)(1) = hom(\{0,1\}, [n]) \text{ is a set of size } \binom{n+1}{2}$$



<u>Eg #1</u>:
$X(0) = \{v_1, v_2\}$
$X(1) = \{e_1, e_2\}$
$X(K) = \emptyset$ for $K \geq 2$.



The idea is that this keeps track of all the gluing information.

$hom([0], [1]) = \{0 \mapsto 0, 0 \mapsto 1\}$.
$X(0 \mapsto 0)(e_i) = v_1$
$X(0 \mapsto 1)(e_i) = v_2$.

$\{0, \dots, n-1\} \longrightarrow \{0, \dots, n\}$
just "skip" an element. We call the map that skips i "d_i."

If X is a Δ-set, define

$$C_\bullet = \{C_n(X) = \mathbb{Z}X(n)\} \qquad \text{(free abelian group)}$$

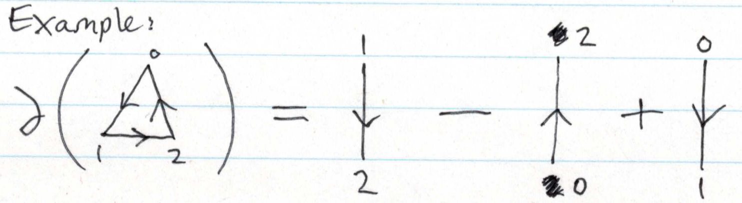
$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

$$X(d_i) : X(n) \longrightarrow X(n-1). \qquad \partial_n = \sum_{i=0}^{n} (-1)^i X(d_i)$$

The idea of d_i is as follows. For example, from $\{0, \dots, 2\} \longrightarrow \{0, \dots, 3\}$,

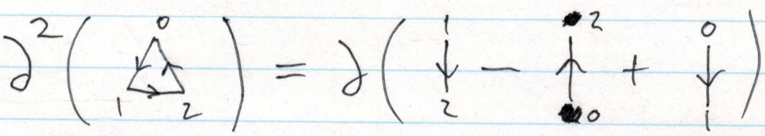
$$d_1 : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 3 \end{array} \qquad \text{(thus 1 is "skipped")},$$

Example:



(formal sum)

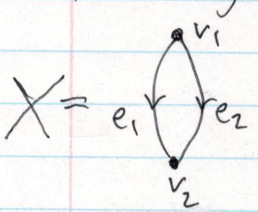
We want to form a chain complex and take its homology groups. In order to have a chain complex, we need $\partial^2 = 0$. We will only check this for the triangle:



$$= (1 - 2) + (2 - 0) + (0 - 1) = 0$$

So "proof by example" $\partial^2 = 0$.

Returning to Eg #1:



$$C_{-1} = 0 \xleftarrow{0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{0} C_2 = 0$$

define: $H_n(X) = H_n(C_\bullet(X)) = \dfrac{\operatorname{Ker} \partial_n}{\operatorname{im} \partial_{n+1}}$.

$$C_0 = \mathbb{Z}v_1 + \mathbb{Z}v_2 \quad \text{and} \quad C_1 = \mathbb{Z}e_1 + \mathbb{Z}e_2.$$

Let's do H_0 first:

$$H_0(X) = \frac{\operatorname{Ker} \partial_0}{\operatorname{im} \partial_1} = \frac{C_0}{\operatorname{im} \partial_1}.$$

Now write

$$[\partial_1] = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

thus

$$\operatorname{im} \partial_1 = \mathbb{Z}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbb{Z}(v_1 - v_2).$$

We conclude

$$H_0(X) = \frac{(\mathbb{Z}v_1 + \mathbb{Z}v_2)}{\mathbb{Z}(v_1 - v_2)} \cong \mathbb{Z}.$$

So it's a free abelian group of rank one.

Let's do the same kind of calculation for H_1.

$$H_1(X) = \frac{\operatorname{Ker} \partial_1}{\operatorname{im} \partial_2} \cong \operatorname{Ker} \partial_1 = \mathbb{Z}(e_1 - e_2)$$

because

$$\partial_1(e_1) = v_1 - v_2 = \partial_1(e_2).$$

Topologically the thing X is a circle. The first homology group is a copy of $\mathbb{Z}$.

H_0 = free abelian group on the set of connected components

H_1 = free abelian group on the set of "1-dimensional holes" in your space.

= abelianisation of the fundamental group.

For finite complexes the groups H_i are always finitely generated. However they're not always free groups as you might guess from this: you could get a $\mathbb{Z}/2\mathbb{Z}$ for example, with the projective plane.

Now we're going to talk about cohomology.
We'll take this chain complex and dualize it.
For each abelian group A we can look at the set of homomorphisms $A \longrightarrow \mathbb{Z}$.
This is like the dual vector space but we're working with abelian groups (i.e. $\mathbb{Z}$-modules).

$$\cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots$$

$$C^n = \mathrm{Hom}(C_n, \mathbb{Z})$$
$$d_n = \partial_n^*.$$

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} C^{n+1} \longrightarrow \cdots$$

Copying/dualizing the complex from last time we get:

$$C^{-1} = 0 \xrightarrow{0} C^0 \xrightarrow{d_1} C^1 \xrightarrow{0} C^2 = 0$$

with

$$C^0 = \mathbb{Z}v_1^* + \mathbb{Z}v_2^* \qquad C^1 = \mathbb{Z}e_1^* + \mathbb{Z}e_2^*$$

Note:

$$[d_1] = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Calculating the zeroth cohomology,

$$H^0(X) = \frac{\mathrm{Ker}\ d_1}{\mathrm{im}\ d_0} \cong \mathrm{Ker}\ d_1 = \mathbb{Z}(v_1^* + v_2^*).$$

Now,

$$H^1(X) = \frac{\mathrm{Ker}\ d_2}{\mathrm{im}\ d_1} = \frac{(\mathbb{Z}e_1^* + \mathbb{Z}e_2^*)}{\mathrm{im}\ d_1}$$

and we have

$$\mathrm{im}\ d_1 = \mathbb{Z}(e_1^* + e_2^*)$$

hence

$$H^1(X) = \frac{(\mathbb{Z}e_1^* + \mathbb{Z}e_2^*)}{\mathbb{Z}(e_1^* + e_2^*)} \qquad \frac{\text{rank 2 thing}}{\text{rank 1 thing}}$$

Compare this to homology we computed. There is a nice kind of symmetry going on here (Poincaré Duality): $H_K \cong H^{n-K}$.

We want to define Čech cohomology. Need a way to get from a topological space to a Δ-complex.

<u>Defn</u>: If C is a category, we construct a Δ-Set out of it called the <u>nerve</u> of that category:

$$N(C)(n) = \{\text{functors from } [n] \text{ to } C\}$$

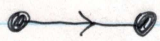
Less abstractly, the functors from $[n]$ to C are just the ways of choosing n composable arrows in C.

$$[n] = 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n.$$

Consider the category

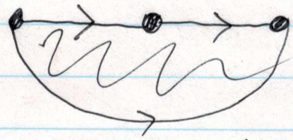
$$C = \begin{tikzcd} X \arrow[r, "f"] & Y \end{tikzcd}$$

then

$N(C) =$ 

For a more complicated example let

$$C = \begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "f \circ g"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

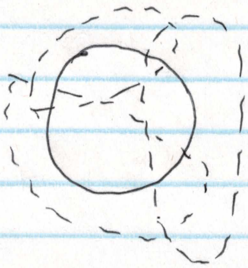
$N(C) =$  (glue a triangle).

<u>Defn</u>: If X is a topological space, and $\mathcal{U}$ is a finite open cover, closed under finite intersections, then $\mathcal{U}$ is a partially ordered set, so it's automatically a category.

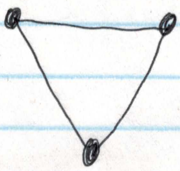
The <u>Čech cohomology</u> of X with respect to $\mathcal{U}$ is the simplicial cohomology of $N(\mathcal{U})$. The Čech cohomology of X is

$$H^n_{\check{C}ech}(X) = \operatorname*{colim}_{\mathcal{U}} H^n(\mathcal{U}).$$

Here's an example. Imagine taking a circle and an open covering that looks like this:



then the nerve of this covering is



There's a theorem that says that you don't really need to take a colimit if all the things in your open cover are contractible.