QQ $X = \operatorname{Spec} R$ (R domain)

Alternate defn of $\mathcal{O}_X$: for $f \in R$,

$$\mathcal{O}_X(U_f) = R_f \quad (= R[f^{-1}])$$

(For $p \in \operatorname{Spec} R$, $\mathcal{O}_{X,p} \simeq R_p$ $(= R[(R-p)^{-1}])$)

~~Defn~~ - For $(X, \mathcal{O})$ a ringed space, $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules if...

Let $M \in \mathrm{Mod}_R$, let $S \subset R$ be mult closed with $1 \in S$

$$M_S = \left\{ \frac{m}{s} \,\middle|\, m \in M,\ s \in S \right\}$$

$$\frac{m}{s} = \frac{n}{r} \quad \text{iff} \quad r \cdot m = s \cdot n$$

(R = domain)

M_S is an R_S-module

Define $\mathcal{F} = \tilde{M}$ on $X = \operatorname{Spec} R$:

$$\mathcal{F}(U_f) = M_f$$

$$(\Rightarrow \mathcal{F}_p \cong M_p)$$

$\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-module

Let $\mathcal{L} \in \mathrm{Mod}\,\mathcal{O}_X$ for $X = \mathrm{Spec}\,R$. ①

$\mathcal{L}$ is <u>qc</u> if $\underset{\substack{\uparrow \\ \mathcal{O}_X\text{-mod}}}{\mathcal{L}} \simeq \tilde{N}$ for $N \in \mathrm{Mod}_R$

Further, $\mathcal{L}$ is <u>c</u> if N is f.g.

Categorically, $\mathrm{Mod}_R \simeq \mathrm{QCoh}_{(X = \mathrm{Spec}\,R,\ \mathcal{O}_X)}$

<u>Ex</u> $X = \mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$, $Z = \{3\} \subset X$

$i : Z \subset X$

$\mathrm{ev}_3 : \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z$

(X = any affine variety, Z = any closed subvariety)

$$i_* \mathcal{O}_Z(U) = \mathcal{O}_Z(U \cap Z) = \begin{Bmatrix} \mathbb{C} & 3 \in U \\ 0 & 3 \notin U \end{Bmatrix}$$

$$\mathrm{ev}_3^U : \mathcal{O}_X(U) \longrightarrow \begin{Bmatrix} \mathbb{C} & 3 \in U \\ 0 & 3 \notin U \end{Bmatrix}$$

$$f \mapsto \begin{Bmatrix} f(3) & 3 \in U \\ 0 & 3 \notin U \end{Bmatrix}$$

$$\mathrm{Ker}(\mathrm{ev}_3)^U = \{ f \in \mathcal{O}_X(U) \mid \mathrm{ev}_3^U(f) = 0 \}$$

$$= \begin{Bmatrix} (x-3)\mathcal{O}_X(U) & 3 \in U \\ \mathcal{O}_X(U) & 3 \notin U \end{Bmatrix}$$

Sheaf of $\mathcal{O}_X$-modules

$\mathcal{O}$ $\ker(ev_3) = \mathcal{I}_{Z\subset X}$ = ideal sheaf of $Z\subset X$ (3)

$\left(\mathcal{I}_{Z\subset X}(U) = \{ f\in \mathcal{O}_X(U) \mid f|_Z = 0 \} \text{ for any } Z\subset X \right)$

$$0 \to \mathcal{I}_{Z\subset X} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

Let $R = \mathbb{C}[t] = \Gamma(\mathbb{A}^1_{\mathbb{C}})$
$I = (t-3)R$

$$0 \to I \to R \to \underset{\underset{\mathbb{C}}{\text{is}}}{R/I} \to 0$$ SES of R-modules

<u>Claim</u> $\widetilde{I} = \mathcal{I}_{Z\subset X}$ $\widetilde{(R/I)} = i_*\mathcal{O}_Z$

For $f\in\mathbb{C}[t]$ $U_f = \{ a\in\mathbb{C} \mid f(a)\neq 0 \}$

$$\widetilde{I}(U_f) = I_f = \left\{ \frac{(t-3)\,p(t)}{f(t)^l} \,\middle|\, p(t)\in\mathbb{C}[t],\ l\geq 0 \right\}$$

$$= (t-3)\,\mathbb{C}[t]_f$$

$$= \begin{cases} (t-3)\mathbb{C}[t]_f & \text{if } f\neq (t-3)^l,\ l\geq 1 \\ \mathbb{C}[t]_f & \text{if } f = (t-3)^l,\ l\geq 1 \end{cases}$$

$$= \begin{cases} (t-3)\mathcal{O}_X(U) & 3\in U \\ \mathcal{O}_X(U) & 3\notin U \end{cases}$$

$$(\widetilde{R/I})(U_f) = (R/I)_f$$

$$= (R/I)[\bar{f}] \quad \text{where} \quad \cancel{\bar f \neq 0 \text{ if } f \nmid t-3} \quad \cancel{\bar f = 0 \text{ if } f \mid t-3}$$

$$= \cancel{R/I} = \mathbb{C}_{f(3)}$$

$$= \begin{cases} \mathbb{C} & f(3) \neq 0 \\ 0 & f(3) = 0 \Leftrightarrow t-3 \mid f(t) \end{cases}$$

$$= \begin{cases} \mathbb{C} & 3 \in U_f \\ 0 & 3 \notin U_f \end{cases}$$

$(X, \mathcal{O})$ scheme, $\mathcal{F}$ an $\mathcal{O}$-module.

$\mathcal{F}$ is <u>quasi coherent</u> if $\exists$ $X = \bigcup_i W_i$ (open)

with $(W_i, \mathcal{O}|_{W_i}) \simeq (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$

$\ni$ $\mathcal{F}|_{W_i} = \widetilde{M_i}$ for some $M_i \in \mathrm{Mod}_{A_i}$

<u>Ex</u> $X = \mathbb{P}^1$, $V = \{[x,1]\}$ $\quad W = \{[1,y]\}$

Let $P = [1,0] \in W$, $P \notin V$

Define $\mathcal{F}(U) = \{f \in k(X)^* \mid \operatorname{div}(f)|_U + P \geq 0\} \cup \{0\}$

$= \{f \in k(X)^* \mid$ on U, f has maybe a pole of order 1 at P, and no other poles

$\cancel{\mathcal{F}(X) = \operatorname{span}_{\mathbb{C}}}$

<u>Claim</u> $\mathcal{F} \in \operatorname{Mod}_{\mathcal{O}_X}$

$h \in \mathcal{O}_X(U),\ f \in \mathcal{F}(U)$

$$\operatorname{div}(hf)|_U + P = \underbrace{\operatorname{div}(h)|_U}_{\geq 0} + \underbrace{\operatorname{div}(f)|_U + P}_{\geq 0} \geq 0$$

$$\mathcal{F}(X) = \operatorname{span}_k\left\{1, \frac{x}{y}\right\} \not\cong_{\mathcal{O}_X(X)} \mathcal{O}_X(X)$$

So $\mathcal{F} \not\cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules

~~ERUR~~) $\mathcal{F}(V)_a = \mathcal{O}_X(V)\ \left(= k[x/y]\right)$

Actually, via id, $\mathcal{F}|_V \underset{\mathcal{O}_X|_V}{\cong} \mathcal{O}_X|_V = \mathcal{O}_{\operatorname{Spec} k[x/y]}$

$$\mathcal{F}(W) = \left\{ \frac{f(x,y)}{y x^{n-1}} \;\middle|\; f(x) \text{ homo } \text{ of deg } n \right\}$$

$$\left(\operatorname{div}\left(\frac{f(x,y)}{y x^{n-1}}\right)\Big|_W + P\right) = \operatorname{div}(f)|_W - P - (n-1)[0,1] + P \geq 0$$

NOT a ring

module over $\mathcal{O}_X(W) = k[y/x]$

$$\mathcal{F}(W) \cong \mathcal{O}_X(W) = k[y/x]$$

as $\mathcal{O}_X(W)$-modules via

$$\frac{f}{y x^{n-1}} \mapsto \frac{f}{x^{n-1}}$$

So $\mathcal{F}|_V \simeq \mathcal{O}_V = \widetilde{k[x/y]}$

$$\mathcal{F}|_W \simeq \mathcal{O}_W = \widetilde{k[y/x]}$$

So $\mathcal{F}$ is coherent!

$M \in \mathrm{Mod}_S$ is <u>S-graded</u> if

$M = \bigoplus_{d \in \mathbb{Z}} M_d \qquad M_d \cap M_e = \cancel{\emptyset}\ 0$

$$S_e \cdot M_d \subset M_{e+d}$$

View $S = k[x,y]$ as a graded module over itself:

$$M = S = \bigoplus_{d \geq -1} M_d \quad \text{where}$$

$$M_d = S_{d+1} \qquad \deg_M(x^n) = n-1$$

M is <u>not</u> a graded ring:

$$M_{e+d} \not\ni M_d \cdot M_e \subset M_{d+e+1}$$

M is a graded S-module

~~Do~~ Mimic $\widetilde{M}$ operation

$\widetilde{M}(U_f) = M_f^{\,0}$

= "degree" 0 elements of M_f $\left(\deg\left(\frac{m}{f^l}\right) = \deg_M(m) - l\deg(f)\right)$

<u>Ex</u> Our M from above

~~Ex~~. $f = 1$

$$k[x,y] = \bigoplus_{d \geq 0} S_d \quad = \text{graded decomp of } k[x,y]$$

$$Y = \operatorname{Proj} k[x,y] = \{ (ax+by) \mid_{a,b \text{ both not zero}}, (0) \}$$

$f \in k[x,y]$ homogeneous, $\mathcal{O}_Y(U_f) = k[x,y]_f^0$

$\quad$ = subring of degree 0 elements of $k[x,y]_f$

<u>Ex</u> $\quad f = x \qquad$ (Intuition: $V(f) = \{[0,1]\}$, so $U_f = \mathbb{P}^1 \setminus \{[0,1]\}$

$$k[x,y]_f = \left\{ \frac{f(x,y)}{x^l} \right\}$$

$2y^2 + 3yx + x^2 \qquad x^2\left(2\left(\frac{y}{x}\right)^2 + 3 \right.$

$$\Rightarrow k[x,y]_f^0 = \left\{ \frac{f(x,y)}{x^l} \;\middle|\; f \text{ homo of deg } l \right\}$$

$$= k[y/x]$$

This ring is a copy of $\mathbb{A}^1$, which is good: $U_f \simeq \mathbb{A}^1$

Also, $M_1 = k$

~~$M \in \mathrm{Mod}_S$ is S-graded~~

So $\mathcal{O}_Y$ is coherent.

$$\mathcal{O}_Y|_{U_x} \simeq \mathcal{O}_{\operatorname{Spec} k[y/x]} \quad \widetilde{k[y/x]}$$

$$\mathcal{O}_Y|_{U_y} \simeq \widetilde{k[x/y]}$$

$\tilde{M}(Y) = \tilde{M}(U_1) = M_0^{0} = S_1 = \mathrm{span}_k\langle x, y\rangle$

<u>$f = x$</u> $\quad \tilde{M}(U_x) = M_x^0 \qquad \deg_M(f) = \deg_S(f) - 1$

$$= \text{degree } 0 \text{ elements of } \left\{ \frac{f(x,y)}{x^l} \right\}$$

$$= \left\{ \frac{f(x,y)}{x^l} \;\middle|\; f \text{ homo}, \deg_S(f) = l+1 \right\}$$

$$= x \left\{ \frac{f(x,y)}{x^{l+1}} \;\middle|\; f \text{ homo}, \deg_S(f) = l+1 \right\}$$

$$= x\, k[y/x]$$

ideal gen by x in $k[y/x]$

$$\Rightarrow \tilde{M}|_{U_x} = \widetilde{(x k[y/x])}$$

$\tilde{M} = \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}(1)$ – <u>locally free:</u>

$$\mathcal{O}(1)|_{U_x} \simeq \mathcal{O}_{\mathbb{P}^1}|_{U_x}$$

$$\mathcal{O}(1)|_{U_y} \simeq \mathcal{O}_{\mathbb{P}^1}|_{U_y}$$

$$\mathcal{O}(1) \not\simeq \mathcal{O}_{\mathbb{P}^1}$$

– shift grading by different amounts to get $\mathcal{O}_{\mathbb{P}^1}(n)$

– actually, $\mathcal{O}(1) \simeq \mathcal{I}(P)$

$\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is invertible if $\exists$ open cover W_α of X so that

$$\mathcal{F}|_{W_\alpha} \simeq \mathcal{O}_X|_{W_\alpha} \text{ as } \mathcal{O}_X|_{W_\alpha}\text{-modules}$$

If $\mathcal{F}, \mathcal{G}$ are invertible, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$

and $\underbrace{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})}$

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \simeq_{\mathcal{O}_X} \mathcal{O}_X$$

$\Rightarrow$ $\underline{\underline{\mathrm{Pic}(X)}}$

$\mathrm{Coh}(X, \mathcal{O}_X), \mathrm{QCoh}(X, \mathcal{O}_X) \in \mathrm{AbCat}$