Computing Fibered Products

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Definition

Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y and let Spec $k(y) \to Y$ be the natural morphism. Then, we define the fiber of the morphism of f over the point y to be the scheme

$$X_y = X \times_Y \text{Spec } k(y)$$

The fibre X_y is a scheme over $k(y)$ and its underlying topological space is homeomorphic to the subset $f^{-1}(y)$ of X.

One can think of the fibre as a family of schemes parameterized by Y.

Computing Fibre Products for Affine Schemes

Recall that if $X = \text{Spec}\ (A)$ and $Y = \text{Spec}\ B$ and if X, Y are over S, then $X \times_S Y = \text{Spec}\ (A \otimes_S B)$.

- Let R be a commutative ring, then we have

$$\begin{aligned} \text{Spec}\ R[x] \otimes_R \text{Spec}\ R[y] &= \text{Spec}\ (R[x] \otimes_R R[y]) \\ &= \text{Spec}\ R[x, y] \end{aligned}$$

- Let $X = \text{Spec}\ \mathbb{Z}/(m)$ and $Y = \text{Spec}\ \mathbb{Z}/(n)$. We have

$$\begin{aligned} X \times_{\mathbb{Z}} Y &= \text{Spec}\ (\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)) \\ &= \text{Spec}\ (\mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z})) \\ &= \text{Spec}\ (\mathbb{Z}/\gcd(m, n)\mathbb{Z}) \end{aligned}$$

If $\gcd(m, n) = 1$ then the tensor product is trivial.

- We would like to compute Spec $\mathbb{C} \times_{\mathbb{R}}$ Spec $\mathbb{C}$. We have

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &= \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/x^2 + 1)) \\
&= (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2 + 1) \\
&= \mathbb{C}[x]/(x^2 + 1) \\
&= \mathbb{C}[x]/((x - i)(x + i)) \\
&= \mathbb{C}/(x - i) \times \mathbb{C}[x]/(x + i) \\
&= \mathbb{C} \times \mathbb{C}
\end{aligned}$$

But we have that Spec $\mathbb{C} \times \mathbb{C} =$ Spec $\mathbb{C} \sqcup$ Spec $\mathbb{C}$. Thus, Spec $\mathbb{C} \times_{\mathbb{R}}$ Spec $\mathbb{C} =$ Spec $\mathbb{C} \sqcup$ Spec $\mathbb{C}$.

Computing the Fibre of a Morphism

Let $X = \text{Spec}\ (\mathbb{Q}[x])$ and $Y = \text{Spec}\ (\mathbb{Q}[Y])$, let $\phi : X \to Y$ by $x \to y^2$ and let f be the induced map from $Y \to X$. Then, the fibre of f at 1 is given by

$$\begin{aligned} Y_1 &= \text{Spec}\ (\mathbb{Q}[y]) \times_{\mathbb{Q}[x]} \text{Spec}\ (\mathbb{Q}[x]) \\ &= \text{Spec}\ (\mathbb{Q}[x,y]/(y^2 - x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1)) \\ &= \text{Spec}\ (\mathbb{Q}[x,y]/(y^2 - x, x - 1)) \\ &= \text{Spec}\ \mathbb{Q}[y]/y^2 - 1 \\ &= \text{Spec}\ \mathbb{Q}[y]/(y-1) \sqcup \mathbb{Q}[y]/(y+1) \end{aligned}$$

Thus, the fibre of 1 is 2 points, as we would expect if were thinking of the fibre as the pullback.

Review

Let R be a ring and let Spec R be the set of prime ideals. Then, we put a topological structure on Spec R by defining for each $S \subset R$

$$V(S) = \{x \in \text{Spec } R : f(x) = 0 \forall f \in S\}$$

and we defined the $V(S)$ to be the closed sets of a topology (and showed that this was indeed a topology).

We then put a Sheaf structure on Spec A. For each $U \subseteq \text{Spec } A$ we define $O(U)$ to be the set of functions $s : U \to \coprod_{p \in U} A_p$ such that $s(p) \in A_p$ and s is locally a quotient of elements of A. That is, for for $p \in U$ there is a neighbourhood V of p with $V \subseteq U$ such that for each $Q \in V$ and $f \notin Q$, $s(Q) = a/f$ in A_Q.

Definition

A Ringed Space is a pair (X, O_X) consisting of a topological space X and a sheaf of rings O_X on X. A morphism of ringed spaces $(X, O_X) \to (Y, O_Y)$ is a pair $(f, f^\#)$ where f is a morphism of topological spaces and $f^\# : O_Y \to f_* O_X$ is a morphism of sheaves.

Definition

A Ringed Space (X, O_X) is called **Locally Ringed** if for each point $p \in X$, the stalk $O_{x,p}$ is a local ring (recall a local ring has exactly 1 maximal ideal). A morphism of Locally Ringed spaces is a morphism of ringed spaces such that the induced map on stalks is a local ring homomorphism (the pullback of the maximal ideal is maximal).

Definition

An Affine Scheme is a locally ringed space (X, O_X) that is isomorphic to Spec A, as a locally ringed space, where A is a commutative ring.

Definition

A Scheme is a locally ringed space (X, O_X) such that for every point p there exists a neighbourhood U such that $(U, O_{X|U})$ is an affine scheme.

Definition

A **Graded-Ring** is a ring S such that we can write $S = \bigoplus_{n \in \mathbb{N}} S_n$ where S_n are abelian groups such that $S_n S_m \subseteq S_{nm}$. If $x \in S_n$ for some n then x is called a **homogeneous element** (with degree n). An ideal I of S is called **homogeneous** if it is generated by homogeneous elements.

Definition

We define $S_+ := \bigoplus_{i>0} S_i \subset S$, which is an ideal and we call this the **Irrelevant-Ideal**.

Definition

A homogeneous prime ideal p is called **relevant** if p does not contain S_+. The Topology of $\text{Proj}(S)$ is given by taking closed sets to be of the form

$$V(I) := \{p : p \text{ is a relevant prime ideal of } S_+ \text{ and } p \subset I\}$$

One can check in the analogous way for affine schemes, that these sets are closed under finite unions and arbitrary intersections and thus do indeed induce a topology. We again call this the **Zariski topology**

Definition

We define a Sheaf of Rings $\mathcal{O}_X$ on Proj(S) by considering, for any open $U \subset X$ all functions

$$s : U \to \bigcup_{p \in U} (S_p)_0$$

such that $s(p) \in (S_p)_0$ and s can be locally represented as a quotient. I.e. for any point $q \in U$ there exists an open neighbourhood V of q and $a, f \in S$ of the same degree, such that for any $p \in V$, $f \notin p$ and $s(p)$ is in the class represented by a/f in $(S_p)_0$. This is almost exactly the same as the Affine Scheme definition but generalized to graded rings and forcing the quotient of "polynomials" to have the same degree in the numerator and denominator.

Theorem

Let k be an algebraically closed field. Then, there is a fully faithful functor t from the category of varieties over k to the category of Schemes over Spec(k). For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the Sheaf of Regular Functions is the restriction of the structure sheaf to the set of closed points.

Definition

A scheme is **Connected** if its topological space is connected. A scheme is **irreducible** if its topological space is connected.

Definition

A scheme is **reduced** if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently, X is reduced if and only if the local rings $\mathcal{O}_p$ for all $p \in X$.

Definition

A scheme X is **integral** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Proposition

A scheme is integral if and only if it is both reduced and irreducible.

Definition

A topological space X is **quasicompact** if any open cover, there exists a finite subcover. This might be one's definition of compactness depending on whether one requires X to be Hausdorff.

Definition

A scheme is **locally Noetherian** if it can be covered by open affine subsets of $\mathrm{Spec}(A_i)$ where each A_i is a Noetherian ring. X is **noetherian** if it is locally noetherian and quasi-compact. Equivalently, X is noetherian, if it can be covered by a finite number of open affine subsets of $\mathrm{Spec}(A_i)$ with each A_i a noetherian ring.

Proposition

A scheme X is locally noetherian if and only if for every Affine subset $U = \mathrm{Spec}\ A$ A is a noetherian ring. In particular, an affine scheme $X = \mathrm{Spec}\ A$ is a notherian scheme if and only if the ring A is a noetherian ring.

Definition

A morphism $f : X \to Y$ of schemes is locally of finite type if there exists a covering of Y by open affine subsets $B_i = \text{Spec } B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \text{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra. The morphism f of **finite type** if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}.

Definition

A morphism $f : X \to Y$ is a **finite morphism** if there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to Spec A_i, where A_i is the B_i-algebra which is a finitely generated B_i-module.

Definition

An **Open Subscheme** of a scheme X is a scheme U whose topological space is an open subset of X and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X.

Definition

An **Open immersion** is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Definition

A **closed Immersion** is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of $sp(Y)$ onto $sp(X)$ and furthermore, the induced map on sheaves $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ on sheaves is surjective.

Definition

A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$. A less abstract way of phrasing this is Y is a closed subscheme of X is $Y \subset X$ and there exists a closed immersion $f : Y \to X$.

Definition

The **Dimension** of a scheme X, denoted dim X is its dimension as a topological space. If Z is an irreducible closed subset of X, then the **codimension** of Z in X denoted by $\text{codim}(Z, X)$ is the supremum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \cdots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\text{codim}(Y, X) = \inf_{Z \subseteq X} \text{codim}(Z, X)$$

where the infimum is taken over all closed subsets of Y.