Fibres of a morphism. Let $f : X \to Y$ be a scheme map. Given $\eta \in Y$, the **fibre** of f over η is defined to be the scheme

$$X_\eta = X \times_Y \operatorname{Spec}(\kappa(\eta))$$

I'm going to explicitly calculate two examples.

Example 1: (Geometry) Let k be an algebraicly closed field with characteristic distinct from 2. Let R be the coordinate ring of two non-parallel lines, say $R = k[x,y]/((x-y)(x+y)) = k[x,y]/(y^2 - x^2)$, and let $S = k[t]$. Consider the k-algebra map $\phi : S \to R$ defined by $\phi(t) = \overline{x}$. Let f denote the corresponding scheme map $(X = \operatorname{Spec} R) \to (Y = \operatorname{Spec} S)$.

I claim that f is the scheme theoretic version of the map $\pi : (V(y^2 - x^2) \subseteq \mathbb{A}^2) \to \mathbb{A}^1$ defined by $(p, \pm p) \mapsto p$. To see this claim, we should calculate f on the closed points of X, ie the maximal ideals $(\overline{x-a}, \pm\overline{y-a})$. f sends this ideal $\mathfrak{q}$ to the ideal $\phi^{-1}(\mathfrak{q})$ of $k[t]$, which is by definition equal to

$$\{p(t) \in k[t] : \overline{p(x)} \in (\overline{x-a}, \pm\overline{y-a})\}$$

and this ideal is of course $(t-a)$. This demonstrates the claim.

Let $\eta = (0)$ be the generic point of Y. Then $\kappa(\eta) = k[t]_{(0)} = k(t)$. It follows that the associated ring is

$$\begin{aligned} k[x,y]/(y^2 - x^2) \otimes_{k[t]} k(t) &= k[x,y]/(y^2 - x^2) \otimes_{k[x]} k(x) \\ &= k(x)[y]/(y^2 - x^2) \\ &= k(x) \times k(x) \end{aligned}$$

The last equality uses the chinese remainder theorem - notice that, inside $k(x)[y]$, we have $(y-x)+(y+x) = k(x)[y]$ since $-2x$ lives inside this ideal, which is a unit of $k(x)$ (char(k) $\neq 2$).

So the fibre X_η is equal to $\operatorname{Spec}(k(x) \times k(x))$. This isn't that surprising, since X is two copies of $\mathbb{A}^1$ which meet at a point, and the function field of $\mathbb{A}^1$ (ie the residue field of the generic point) is $k(x)$. Notice that it has two points, and that $\pi^{-1}(\mathbb{A}^1)$ also has two irreducible components - this is no coincidence.

Now let $\mathfrak{p} = (t-a)$ be a closed point of Y. Then the residue field is $\kappa(\mathfrak{p}) = k[t]/(t-a)$ (this is of course isomorphic to k, but we care about it's $k[t]$ algebra structure, which we still need to keep track of). The algebra associated to our fibre is

$$\begin{aligned} k[x,y]/(y^2 - x^2) \otimes_{k[t]} k[t]/(t-a) &= k[x,y]/(y^2 - x^2) \otimes_{k[x]} k[x]/(x-a) \\ &= k[a,y]/(y^2 - a^2) \\ &= k[y]/(y^2 - a^2) \end{aligned}$$

If $a \neq 0$, this ring is equal to $k \times k$ by the chinese remainder theorem (this also uses that char $k \neq 2$). Notice that the spectrum of this space has two points, which is equal to $\#\pi^{-1}(a)$. If $a = 0$, this ring is equal to $k[y]/(y^2)$, whose spectrum has one point - the ideal $(\overline{y})$ - and that the fibre $\pi^{-1}(0)$ also has exactly one point.

What do we gain from this example? I think it is pretty intuitive that for $a \neq 0$, the fibre $\pi^{-1}(a)$ should be assigned the ring $k \times k$. However, we weren't really sure what ring should be assigned to $\pi^{-1}(0)$ - it didn't seem quite right to just assign it k, since that doesn't seem to take into account the fact that $(0,0)$ is the meeting point of our two lines. Also notice that each ring we got is a two-dimensional vector space over $\kappa(\eta)$ - this is also no coincidence - the map π intuitively has 'degree 2'.

Example 2: (Arithmetic) Let $X = \operatorname{Spec} \mathbb{Z}[x]$ and let $Y = \operatorname{Spec} \mathbb{Z}$. and let f be the (only) morphism $X \to Y$ - this corresponds to the (only) ring map $\mathbb{Z} \to \mathbb{Z}[x]$. I'm first going to describe the set X - it has four types of points:

- The ideal (0), which we denote by η - the generic point of X.
- The ideal (p) for a prime number p.
- The ideal $(f(x))$ for $f(x)$ irreducible over $\mathbb{Z}$.
- The ideal $(f(x), p)$ for p prime and $f(x)$ irreducible over $\mathbb{Z}_p$ (meaning $\mathbb{Z}/p\mathbb{Z}$).

An aside : that I am able to explicitly describe X takes advantage of the fact that $\dim \mathbb{Z}[x] = 2$ (and that $\mathbb{Z}[x]$ is a UFD). For example, I don't think you can write down an analagous list for $\mathbb{Z}[x, y]$ (or $k[x, y, z]$).

Let's first describe the set-theoretic fibres of f. Above $(0) \in \operatorname{Spec} \mathbb{Z}$ we have the ideal from bullet 1, and every ideal from bullet 3. Above $(p) \in \operatorname{Spec} \mathbb{Z}$ we have the ideal (p) from bullet 2 and every prime ideal from bullet 4 (for our fixed p).

What are the residue fields of Y? No problem: $\kappa((0)) = \mathbb{Q}$, and $\kappa((p)) = \mathbb{Z}_p$. So the associated rings are

$$\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[x]$$

$$\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p[x]$$

So $X_{(0)} = \operatorname{Spec} \mathbb{Q}[x]$ - notice that this set naturally bijects our set theoretic fibre. Similarly, $X_{(p)} = \operatorname{Spec} \mathbb{Z}_p[x]$ which bijects our fibre. Both these rings seem like pretty reasonable fibres for our map!