Properties of Schemes II

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Overview

Review

Last time we defined what it means for a scheme to be *connected*, *irreducible*, *reduced* and *integral*. We saw that a scheme is integral if and only if it is both reduced and irreducible.

Next, we defined *locally noetherian* schemes to be those which can be covered by open affine subsets Spec A_i with each ring A_i noetherian. We called a scheme *noetherian* if it is locally noetherian and *quasi-compact* (every open cover has a finite subcover).

We then discussed morphisms of *locally finite type*, *finite type*, and *finite* morphisms, finally moving on to define *immersions* and *subschemes*.

Eeshan didn't quite finish on Thursday, so I'll pick up where he left off.

Characterisation of locally noetherian schemes

We will now prove a proposition which was stated without proof last time.

Proposition 3.2

A scheme X is locally noetherian if and only if for every open affine subset $U = \text{Spec } A$, A is a noetherian ring. In particular, an affine scheme $X = \text{Spec } A$ is a noetherian scheme if and only if the ring A is a noetherian ring.

First, we need the following lemma.

Lemma

Let A be a ring, and $f_1, \ldots, f_r \in A$ generate the unit ideal (i.e. A). Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\varphi_i : A \to A_{f_i}$ be the localisation map, $i = 1, \ldots, r$. Then

$$\mathfrak{a} = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i}).$$

Proof of Lemma

The inclusion $\subseteq$ is obvious. We prove $\supseteq$. Given an element $b \in A$ contained in this intersection, we can write $\varphi_i(b) = a_i/f_i^{n_i}$ in A_{f_i} for each i, where $a_i \in \mathfrak{a}$ and $n_i > 0$. Increasing the n_i if necessary, we can make them all equal to a fixed n. This means that in A we have

$$f_i^{m_i}(f_i^n b - a_i) = 0$$

for some m_i. As before, we can make all the $m_i = m$. Thus $f_i^{m+n} b \in \mathfrak{a}$ for each i. Since $f_1, \ldots, f_r$ generate the unit ideal, the same is true of their Nth powers for any N. Take $N = n + m$. Then we have $1 = \sum c_i f_i^N$ for suitable $c_i \in A$. Hence

$$b = \sum c_i f_i^N b \in \mathfrak{a}$$

as required.

We now proceed to the proof of the proposition.

Proof of Proposition

The "if" direction is clear. We must show that if X is locally noetherian, and if $U = \text{Spec } A$ is an open affine subset, then A is a noetherian ring.

First, note that if B is a noetherian ring, so is any localisation B_f. The open subsets $D(f) \cong \text{Spec } B_f$ form a base for the topology of Spec B. Hence on a locally noetherian scheme X there is a base for the topology consisting of the spectra of noetherian rings. In particular, our open set U can be covered by spectra of noetherian rings.

So we have reduced to proving the following: let $X = \text{Spec } A$ be an affine scheme, which can be covered by open subsets which are spectra of noetherian rings. Then A is noetherian. Let $U = \text{Spec } B$ be an open subset of X, with B noetherian. Then for some $f \in A$, $D(f) \subseteq U$. Let $\overline{f}$ be the image of f in B. Then $A_f \cong B_{\overline{f}}$, hence A_f is noetherian. So we can cover X by open subsets $D(f) \cong \text{Spec } A_f$ with A_f noetherian. Since X is quasi-compact, a finite number will do.

Proof of Proposition

So now we have reduced to a purely algebraic problem: A is a ring, $f_1, \dots, f_r$ are a finite number of elements of A, which generate the unit ideal, and each localisation A_{f_i} is noetherian. We have to show A is noetherian. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be an ascending chain of ideals in A. Then for each i,

$$\varphi_i(\mathfrak{a}_1) \cdot A_{f_i} \subseteq \varphi_i(\mathfrak{a}_2) \cdot A_{f_i} \subseteq \dots$$

is an ascending chain of ideals in A_{f_i}, which stabilises since A_{f_i} is noetherian. There are only finitely many A_{f_i}, so from the lemma we conclude the original chain is eventually stationary, so A is noetherian.

Dimension of a scheme

Definition

The **dimension** of a scheme X, denoted $\dim X$, is its dimension as a topological space: the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X.

If Z is an irreducible closed subset of X, then the **codimension** of Z in X, denoted $\operatorname{codim}(Z, X)$, is the supremum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \ldots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf\{\operatorname{codim}(Z, X) : Z \subseteq Y \text{ closed irreducible}\}.$$

Example

If $X = \text{Spec } A$ is an affine scheme, then the dimension of X is the same as the Krull dimension of A.

Warning

The concepts of dimension and codimension may not behave so well over arbitrary schemes. Our intuition is derived from working with schemes of finite type over a field, where these notions are well-behaved. For example, if X is an affine integral scheme of finite type over a field k, and if $Y \subseteq X$ is any closed irreducible subset, then by a result in Chapter 1, $\dim\, Y + \text{codim}(Y, X) = \dim\, X$. But on arbitrary (even noetherian) schemes, funny things can happen.

Fibred product

Definition

Let S be a scheme, and let X, Y be schemes over S (i.e. schemes with morphisms to S). We define the **fibred product** of X and Y over S, denoted $X \times_S Y$, to be a scheme, together with morphisms p_1 and p_2 to X and Y respectively, which make the obvious diagram commute, such that $X \times_S Y$ is terminal among all such cones.

Proposition

For any two schemes X and Y over a scheme S, the fibred product $X \times_S Y$ exists, and is unique up to unique isomorphism.

- Uniqueness is a consequence of abstract nonsense, so we need only show existence.
- If all the schemes are affine, say $X = \text{Spec } A$, $Y = \text{Spec } B$, and $S = \text{Spec } R$, then $\text{Spec } A \otimes_R B$ will serve as $X \times_S Y$. This is not hard to see: a morphism $Z \to \text{Spec } A \otimes_R B$ is the same as a ring map $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$, which is the same as giving ring maps $A, B \to \Gamma(Z, \mathcal{O}_Z)$ which induce the same map on R. This in turn is the same as giving maps $Z \to X, Y$ which cause the obvious diagram to commute.
- If X and Y are schemes, X open-covered by $\{U_i\}$, then giving a map $f : X \to Y$ is the same thing as giving a compatible family of maps $f_i : U_i \to Y$.

- If X, Y are schemes over S, and $U \subseteq X$ is an open subset, and if $X \times_S Y$ exists, then $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U with Y over S.
- *Hard part*: if X, Y are schemes over S, $\{X_i\}$ is an open covering of X, and for each i, $X_i \times_S Y$ exists, then $X \times_S Y$ exists.
- We know from the beginning that if X, Y, S are all affine then $X \times_S Y$ exists. Thus we conclude that for any X, but Y, S affine, the product $X \times_S Y$ exists. Doing the same thing again but interchanging X and Y tells us that $X \times_S Y$ exists for *any* X and Y over an affine S.
- *Final step*: for arbitrary X, Y, S, let $q : X \to S$ and $r : Y \to S$ be the given morphisms, and $\{S_i\}$ be an open affine cover of S. Put $X_i = q^{-1}(S_i)$ and $Y_i = r^{-1}(S_i)$. Then by the previous step $X_i \times_{S_i} Y_i$ exists. One can show this same scheme is a product for X_i and Y over S. But then $X \times_S Y$ exists, completing the proof.

Fibres of a morphism

Here is a use of fibred products.

Definition

Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y and let Spec $k(y) \to Y$ be the natural morphism. Then we define the **fibre** of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

The fibre X_y is a scheme over $k(y)$, and one can show that its underlying topological space is homeomorphic to the subset $f^{-1}(y)$ of X.

So we can regard a morphism as a "family of schemes" (i.e. its fibres) parametrized by the points of the image scheme.