I wrote these pretty quickly, so not sure if they say anything useful or not.

Let $(X, \mathcal{O})$ be a scheme. We want to define what a closed subscheme is. Your first guess might to be say that it is a closed subset Z of X, along with the sheaf $j^{-1}\mathcal{O}$ where $j : Z \subseteq X$ is the inclusion. However, there are many different sheaves we may wish to put on the subset - that's the issue!

This definition doesn't really have a good analogue in the land of varieties over an algebraicly closed field - we are actually getting a legitimate generalization here. I think this definition is best motivated by recalling how, when talking about varieties over algebraicly closed fields, the operation $I \mapsto V(I)$ for an ideal I of $k[x]$ loses information, ie it isn't 1-1. For example, $V(x^2) = V(x)$ while the ideal (x^2) and (x) are totally different creatures - we want to imagine (x) as the regular functions which vanish at the origin to order at least 1, while (x^2) is the functions which vanish along the origin to order at least 2.

Let R be the ring $k[x]/(x)$ and let S be the ring $k[x]/(x^2)$. Then $X = \operatorname{Spec} R = \{(0)\}$ and $Y = \operatorname{Spec} S = \{(x)\}$ (S isn't a domain, so the zero ideal isn't prime). When I write $\mathcal{O}_R$, I mean $\mathcal{O}_{\operatorname{Spec} R}$, and likewise for $\mathcal{O}_S$. The topological space of maximal ideals of X and Y are homeomorphic, which is what we expect - both spaces only have closed points, anc closed 'points' on a scheme are our usual points on a variety, and $V(x) = V(x^2)$. However, the structure sheaves are totally different, which is what we want to be true.

Both $(X, \mathcal{O}_R)$ and $(Y, \mathcal{O}_S)$ are closed subschemes of $Z = \operatorname{Spec} k[t]$: let $i : X \to Z$ be defined by $(0) \mapsto (t)$ and let $j : Y \to Z$ be defined by $(x) \mapsto (t)$ (these two maps correspond to the ring maps $k[t] \to k[x]/(x)$ and $k[t] \to k[x]/(x^2)$) with their spaces being the same closed zubset of Z. You can then check that in both cases, we have that $\mathcal{O}_Z \to i_*\mathcal{O}_R$ and $\mathcal{O}_Z \to j_*\mathcal{O}_S$ are surjective, as required for the definition of a closed immersion. (Reality Check: Why is $\mathcal{O}_Z \to i_*\mathcal{O}_R$ surjective? You can either verify this for the map on global sections, which is the surjection $k[x] \to k[x]/(x)$ or on the stalks where you get the surjection $k[x]_{(0)} \to (k[x]/(x))_{(\bar{0})}$).

Note that even though we have the same closed subspace, these are not the same closed subschemes - you certainly can't find an isomorphism from X to Y commuting with i and j, because there is no isomorphism from X to Y.