Properties of Schemes

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Definition

A scheme is **Connected** if its topological space is connected. A scheme is **irreducible** if its topological space is connected.

Definition

A scheme is **reduced** if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently, X is reduced if and only if the local rings $\mathcal{O}_p$ for all $p \in X$.

Definition

A scheme X is **integral** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Proposition

A scheme is integral if and only if it is both reduced and irreducible.

proof

If our scheme is integral, then since each $\mathcal{O}_X(U)$ is an integral domain, it has no nilpotent elements and so it is reduced. If X is not irreducible then there exist disjoint open sets U_1, U_2. Then, by the restriction and gluing axiom for Sheaves, it follows that $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$. For example, the map going right, take the restriction of $s \in O(U_1 \cup U_2)$ to $O(U_1)$ and $O(U_2)$. For the reverse direction use gluing and identity to get s back. But note that $\mathcal{O}(U_1) \times \mathcal{O}(U_2)$ is not an integral domain (just take $(s, 0) \cdot (0, s)$). Thus, integral implies irreducible.

Now suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset and let $f, g \in \mathcal{O}(U)$ such that $fg = 0$. We wish to show either $f = 0$ or $g = 0$.

Proof Continued

Let $Y = \{x \in U : f_x \in m_x\}$ and $Z = \{x \in U : g_x \in m_x\}$. Then, Y, Z are closed subsets of U. To check this, let V be an affine subset of U, let X_f denote the points of U such that $f_x \notin m_x$ and show that $X_f = U \cap X_f = D(f)$ and thus conclude that Y, Z are closed. Moreover, we have that $Y \cup Z = U$. This is because at each x we have $(fg)_x = 0 \iff f_x g_x \in m_x$ and m_x is prime so either $f_x \in m_x$ or $g_x \in m_x$. Thus, either $U = Y$ or $U = Z$, we may assume $U = Y$. Now consider $V \subset U$ affine and f, g restricted to U. For each $v \in V$ we have $f_v \in m_v$ and this means precisely that $f(v) = 0$ i.e. $f \in v$ (where v is a prime ideal since we restricted ourself down to an affine scheme). Thus, for all every prime ideal $v \in V$, $f \in V$ and so f is in the nilradical and so f is nilpotent. But, we assumed that X was reduced and has no nilpotent elements, therefore $f = 0$ and so X is integral.

Definition

A topological space X is **quasicompact** if any open cover, there exists a finite subcover. This might be one's definition of compactness depending on whether one requires X to be Hausdorff.

Definition

A scheme is **locally Noetherian** if it can be covered by open affine subsets of Spec(A_i) where each A_i is a Noetherian ring. X is **noetherian** if it is locally noetherian and quasi-compact. Equivalently, X is noetherian, if it can be covered by a finite number of open affine subsets of Spec(A_i) with each A_i a noetherian ring.

Remark

If X is a Noetherian scheme, then its underlying topological space is noetherian, however, if if you have a scheme with underlying space is Noetherian, the scheme does not have to be Noetherian.

Proposition

A scheme X is locally noetherian if and only if for every open affine subset $U = \text{Spec } A$, A is a noetherian ring. In particular, an affine scheme $X = \text{Spec } A$ is a noetherian scheme if and only if the ring A is a noetherian ring.

proof

Proof next time.

Definition

A morphism $f : X \to Y$ of schemes is locally of finite type if there exists a covering of Y by open affine subsets $B_i = \text{Spec } B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \text{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra. The morphism f of **finite type** if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}.

Definition

A morphism $f : X \to Y$ is a **finite morphism** if there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to Spec A_i, where A_i is the B_i-algebra which is a finitely generated B_i-module.

Example

If V is an affine variety over k (ACF) then recall $t(V)$ is the associated scheme over Spec k. So there exists a morphism

$$f : t(V) \to \text{Spec } k$$

Well note that $t(V)$ is affine so it is of finite type because firstly it admits a trivial cover by affine schemes. Secondly, $t(V) \cong X$ where $X = \text{Spec } A$ where A is the coordinate ring of V and the coordinate ring of a variety is a finitely generated k-algebra. Moreover, the coordinate ring is Noetherian, so $t(V)$ is noetherian and similarly also integral.

Definition

An **Open Subscheme** of a scheme X is a scheme U whose topological space is an open subset of X and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X.

Definition

An **Open immersion** is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Definition

A **closed Immersion** is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of $sp(Y)$ onto $sp(X)$ and furthermore, the induced map on sheaves $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ on sheaves is surjective.

Definition

A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$. A less abstract way of phrasing this is Y is a closed subscheme of X is $Y \subset X$ and there exists a closed immersion $f : Y \to X$.

Example

Let A be a ring and let a be an ideal of A. Let $X = \text{Spec } A$ and let $Y = \text{Spec } A/a$. Then the quotient map $\pi : A \to A/a$ induces a morphism of schemes $f : Y \to X$ given by $f(p) = \pi^{-1}(p)$. (By the fourth iso. thm?) it follows that f is a homeomorphism onto the closed subset $V(a)$ of X. By an exercise in Hartshorne, this also tells you that induced map on stalks is surjective and thus the map on sheaves is surjective. Thus, f is a closed immersion. Hartshorne says that given a closed subset Y there are many different closed subscheme structures on Y and these correspond to the ideals a such that $V(a) = Y$. I suppose what this means is we have $V(a) = Y = V(b)$, then we get different maps $f, f' : Y \to X$ which are not equivalent under the prescribed relation. Also apparently, every subscheme structure on Y arises in this way.

Example

Let $A = k[x, y]$ where k is a field. Then, Spec $A = \mathbb{A}_k^2$ is the affine plane over k. The ideal $I = (xy) = ((x-0)(y-0))$ is a reducible subscheme and can be written as the union of the X and Y axis. The ideal (x^2) gives a subscheme structure with nilpotents on the y-axis?

Example

Let V be an affine variety over the field k and let W be a closed subvariety. We have that W corresponds to a prime ideal p in the coordinate ring A of V. Let $X = t(V)$ and $Y = t(W)$ be the associated schemes. Then, let $X = \text{Spec } A$ and Y is a closed subscheme "defined" by p (defined as in the previous slide). For each $n \geq 1$ let Y_n denote the subscheme corresponding to the ideal p^n (I think this is just $V(p^n)$), then $Y_1 = Y$ and for $n > 1$, Y_n is a nonreduced scheme structure which does not correspond to any subvariety of V. Y_n is called the **infinitessimal neighbourhood** of Y in X. Later on we will see what happens as we take $n \to \infty$.

Example

Let X be a scheme and Y a closed subset. We can define many different closed subscheme structures on Y but there is one that is "smaller" than the others and called the **reduced induced closed subscheme structure**. First let $X = \text{Spec } A$ and let $a \subseteq A$ be the ideal obtained by intersection all ideals in Y. This is apparently the largest ideal for which $V(a) = Y$ and so defined the reduced induced structure to be the one defined by $V(a)$. The case for arbitrary schemes takes more work, but the idea is for each affine subset U_i of X define $Y_i = Y \cap U_i$ and put the induced structure that we had on affine schemes on Y_i. Glue the sheaves together to get a sheaf on Y and get an induced structure on Y.

Definition

The **Dimension** of a scheme X, denoted dim X is its dimension as a topological space. If Z is an irreducible closed subset of X, then the **codimension** of Z in X denoted by $\operatorname{codim}(Z, X)$ is the supremum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \cdots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq X} \operatorname{codim}(Z, X)$$

where the infimum is taken over all closed subsets of Y.

Example

If $X = \text{Spec A}$ is an affine scheme, then the dimension of X is the same as the Krull Dimension.

One needs to be careful when applying the definitions of Dimension and Codimension. Hartshorne says that our intuition of dimension really comes form working with schemes of finite type over a field (for example varieties). For example, if X is an integral affine scheme of finite type over a field k and if $Y \subseteq X$ is a closed irreducible subset, then we have

$$\dim(Y) + \operatorname{codim}(Y, X) = \dim(X)$$

But apparently this does not hold in the case of arbitrary schemes or even Noetherian Schemes.

References

R. Hartshorne, "Algebraic Geometry; Section 2.3"