Projective Schemes

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From Last Time

Definition

A **Graded-Ring** is a ring S such that we can write $S = \bigoplus_{n \in \mathbb{N}} S_n$ where S_n are abelian groups such that $S_n S_m \subseteq S_{nm}$. If $x \in S_n$ for some n then x is called a **homogeneous element** (with degree n). An ideal I of S is called **homogeneous** if it is generated by homogeneous elements.

Definition

If A is a ring such that $S_0 = A$, then we say S is a graded ring over A and that A is the **Base Ring**. For example, let $S = A[x_0, \ldots, x_n]$ with the standard gradition, then A is the base ring.

Definition

We define $S_+ := \bigoplus_{i>0} S_i \subset S$, which is an ideal and we call this the **Irrelevant-Ideal**.

Given a graded ring S, we will construct a Projective Scheme on S, which we denote by Proj(S), using the same steps as we did for Affine Schemes. We first talk about the set S, we then proceed to put a topology on S, and then put a locally ringed structure. The way to construct it will be using Affine Schemes as building blocks.

Definition

A homogeneous prime ideal p is called **relevant** if p does not contain S_+. The Topology of Proj(S) is given by taking closed sets to be of the form

$$V(I) := \{p : p \text{ is a relevant prime ideal of } S_+ \text{ and } p \subset I\}$$

One can check in the analogous way for affine schemes, that these sets are closed under finite unions and arbitrary intersections and thus do indeed induce a topology. We again call this the **Zariski topology**

Let $f \in S_+$ be homogeneous, and note that localization $S_f = S[f^{-1}]$ is a naturally $\mathbb{Z}$-graded ring (i.e. instead of indexing over $\mathbb{N}$ in the direct sum, we now index over $\mathbb{Z}$). Let $(S_f)_0$ denote the 0-graded part of the graded ring S_f. We define a sheaf structure on Proj(S) in an analogous way for Affine Schemes.

Definition

We define a Sheaf of Rings $\mathcal{O}_X$ on Proj(S) by considering, for any open $U \subset X$ all functions

$$s : U \to \bigcup_{p \in U} (S_p)_0$$

such that $s(p) \in (S_p)_0$ and s can be locally represented as a quotient. I.e. for any point $q \in U$ there exists an open neighbourhood V of q and $a, f \in S$ of the same degree, such that for any $p \in V$, $f \notin p$ and $s(p)$ is in the class represented by a/f in $(S_p)_0$. This is almost exactly the same as the Affine Scheme definition but generalized to graded rings and forcing the quotient of "polynomials" to have the same degree in the numerator and denominator.

Proposition

Let S be a graded ring then

1. For any $p \in \text{Proj} S$, the stalk $\mathcal{O}_p$ is isomorphic to the local ring $S_{(p)}$.
2. let $X = \text{Proj}(S)$. For any homogeneous element $f \in S_+$, let

$$U_f = \{p \in \text{Proj}(S) | f \notin p\}$$

then U_f is open in $\text{Proj}(S)$. Moreover, these sets cover X and we have an isomorphism of locally ringed spaces $(U_f, \mathcal{O}_X|_{U_f}) \cong \text{Spec}(S_f)_0$.

In particular, $\text{Proj}(S)$ is a scheme.

Proof

The proof of (1) follows in the same way as the proof that for affine schemes, for $p \in \text{Spec}(A)$, the stalk $\mathcal{O}_p \cong A_p$. Next, by definition, we have $U_f = D(f)$ and so U_f is open and also form a cover of X. So fix homogeneous $f \in S_+$.

Proof Continued

Now we define an isomorphism of locally ringed spaces

$$(\varphi, \varphi^{\#}) : (U_f, \mathcal{O}_X|_{U_f}) \to \text{Spec}S_{(f)}$$

Let a be a homogeneous ideal of S, let $\varphi(a) = aS_p \cap S_{(p)}$. In particular, if a is prime, then $\varphi(a)$ is a prime ideal of $S_{(f)}$. Apparently from the properties of localization it follows that φ is a bijection. Furthermore, if $p \in U_f$, then $p \supseteq a \iff \varphi(p) \supseteq \varphi(a)$ so φ is a homeomorphism. Moreover, $S_{(p)}$ and $(S_{(f)})_{\varphi(p)}$ are naturally isomorphic as local rings and the homeomorphism and morphism of local rings induces an isomorphism of Sheaves. Hence, $(\varphi, \varphi^{\#}$ is an isomorphism of locally ringed spaced and so U_f is an affine scheme. Therefore, Proj(S) is indeed a scheme.

Definition

Let R be a ring, then we define **Projective n-space** over R, as $\mathbb{P}^n_R := \text{Proj}(R[x_1, \ldots, x_n])$. In particular, if k is algebraically closed, then $\mathbb{P}^k_n$ is a scheme whose subspace of closed points is naturally homeomorphic to the variety called Projective space.

An equivalent way to define Projective space would be to generalize the construction of Projective Varieties. We could construct it by "gluing" together lines through the origin. I.e. Let $U_i := \text{Spec}(R[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ and then glue these U_i together (I think in a similar fashion to the projective line example from last week). From this construction it is much more clear that it is a scheme because we constructed it by gluing together affine schemes. However, it seems like the Proj construction is preferred as it is more general and does not require any choices to be made. However, it is unclear, at least to me at the moment, the equivalence of these definitions.

Next, we wish to show that a scheme does generalize the notion of a variety. We saw in the example of the affine spaces, that varieties aren't quite schemes. For example the affine line scheme has more points than the affine variety.

Definition

Fix a Scheme S. A Scheme over S is a scheme X wish a morphism $X \to S$.

If X is a topological space, then let $t(X)$ be the set of irreducible closed subsets of X. Then, $t(X)$ is naturally a topological space and if we define a map $\alpha : X \to t(X)$ by sending a point to its closure then α induces a bijection the closed sets of X and $t(X)$. Moreover, observe that

1. If $Y \subset X$ is a closed subset then $t(Y) \subset t(X)$
2. If Y_1, Y_2 are two closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$
3. If Y_α is any collection of closed subsets, then $t(\cap Y_\alpha) = \cap t(Y_\alpha)$.

Thus, $t(\alpha)$ defines a topology on $t(X)$.

Theorem

Let k be an algebraically closed field. Then, there is a fully faithful functor t from the category of varieties over k to the category of Schemes over $\mathrm{Spec}(k)$. For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the Sheaf of Regular Functions is the restriction of the structure sheaf to the set of closed points.

Proof

Let k be an algebraically closed field, V a variety over k and $\mathcal{O}_V$ be the sheaf of regular functions. We aim to show that $(t(V), \alpha_*(\mathcal{O}_V))$ is a scheme. Any variety has a cover by affine subvarieties, so it suffices to consider the case when V is affine with coordinate ring A. Let $X = \mathrm{Spec}(A)$. We define a morphism of locally ringed spaced

$$\beta(f, f^{\#}) : (V, \mathcal{O}_V) \to f_*\mathcal{O}_V(U) = \mathcal{O}_V(f^{-1}(U))$$

Proof Continued

If $u \in V$, let $f(u) = m_u$, where m_u is the maximal ideal of elements of A vanishing at u. Then, f induces a bijection between the closed points of X (namely maximal ideals) and the points of V. Thus, f is also a homeomorphism onto its image. Now let $U \subset X$ be open. We aim to define a ring homomorphism

$$f^{\#}(U) : \mathcal{O}_X(U) \to \beta_* \mathcal{O}_V(U) = \mathcal{O}_V(\beta^{-1}(U))$$

Let $s \in \mathcal{O}_X(U)$ and given $p \in B^{-1}(U)$, we define $r(p)$ by taking the image of s in the stalk $\mathcal{O}_{X,\beta(p)} \cong A_{m_p}$. Have

$$A_{m_p}/m_p \cong k$$

Thus, r gives a function from $\beta^{-1}(U) \to k$. r is regular and is a local ring isomorphism $\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}(U))$. Since the prime ideals of A are in 1-1 correspondence with irreducible closed subsets of V we have $(X, \mathcal{O}_X) \cong (t(V), \alpha_* \mathcal{O}_V)$.

Proof Continued

Let $I : t(V) \to X$ be the map which sends an irreducible closed subset to its ideal. We already have I is a homeomorphism. We also have $I_*(\alpha_* \mathcal{O}_V) = \beta_* \mathcal{O}_V$. Hence, the (I, I_*) is an isomorphism of schemes so $t(V)$ is an affine scheme.