The Joy of Schemes

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From Last Time

Definition

Let R be a commutative ring. The *Spectrum* of R, denoted by Spec R, is the set of prime ideals $P \subseteq R$ (Though we also put a topology and Sheaf structure on it).

Definition

The elements of our ring R can be viewed as a function on Spec R and the value of $r \in R$ at a prime ideal $[p]$ is written as $r \mod (p)$. We can even consider "Rational" functions, take $\frac{27}{4}$. This has a pole at (2) and a triple root at (3). Its value at (5) is given by

$$27 \times 4 \equiv 2 \times (-1) \equiv 3 \mod (5)$$

The Topology

We put a topology on the Spectrum in a vaguely analogous way to the Zariski Topology on varieties. We define

Definition

For $S \subset R$ we define

$$V(S) = \{x \in \text{Spec } R : f(x) = 0 \text{ for all } f \in S\}$$

and define these sets to be Closed and look at the topology they generate. We proved last time that this does in fact generate a topology by checking closure under finite unions and arbitrary intersections.

Sheaf Structure

We also defined a sheaf on Spec$A$, called the Structure Sheaf, denoted by $\mathcal{O}$ and was defined as follows. $A_{\mathcal{P}}$ denotes the localisation of A at $\mathcal{P}$, that is, $(A \setminus \mathcal{P})^{-1}A$.

Definition

For each open $U \subseteq \text{Spec } A$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathcal{P} \in U} A_{\mathcal{P}}$ such that $s(\mathcal{P}) \in A_{\mathcal{P}}$ for each $\mathcal{P}$, and s is "locally a quotient of elements of A", that is, for each $\mathcal{P} \in U$, there is a neighbourhood V of $\mathcal{P}$, with $V \subseteq U$, and $a, f \in A$ such that for each $\mathcal{Q} \in V$, $f \notin \mathcal{Q}$, and $s(\mathcal{Q}) = a/f$ in $A_{\mathcal{Q}}$.

This is similar to the definition of regular functions on a variety, except instead of considering functions to a field, we look at functions into the various local rings.

Ringed Spaces

To each commutative ring A, we have now associated its spectrum $(\operatorname{Spec} A, \mathcal{O})$. We want to get a functor out of this, so we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is that of a *locally ringed space*. Recall the concept of a *direct image sheaf* $f_*\mathcal{F}$.

Definition

A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A **morphism** of ringed spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair $(f, f^\#)$ consisting of a continuous map $f : X \to Y$ and a morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

Motivation

Consider once again the case of Differentiable manifolds. A differentiable manifold is defined to be something obtained by gluing together these open balls in a way that maintains that the transition functions on the overlap are smooth. Equivalently, we could also define a manifold in the language of Sheaves. We could define it as a topological space X together with a sheaf of rings. We would require an open cover of X by open sets such that the ringed space associated to each open set was isomorphic to a ball in $\mathbb{R}^n$ equipped with the Sheaf of differentiable functions. In this definition, the open sets are the basic patches we use to construct the set. In Algebraic Geometry, these patches will be the notion of an Affine Scheme.

Definition

A Local Ring is a ring R that has exactly one maximal ideal. Equivalently, the sum of two non-units is a non unit.

Consider coninuous function on the real line and consider the germ of continuous functions at 0. A germ is invertible if and only if $f(0) \neq 0$. So through this characterization, one sees that the sum any two non-invertible germs is once again not invertible. The maximal ideal consists precisely of the germs such that $f(0) = 0$.

Definition

The Ringed space (X, O_X) is called *Locally Ringed* if for each point $p \in X$, the stalk $C_{x,p}$ is a local ring.

Recall that Stalk at a point does have a ring structure where one can add/multiply equivalence classes.

A Morphism of Locally Ringed Spaces is a morphism of ringed spaces $(f, f^{\#})$ such that for each $p \in X$ the induced map of local rings

$$f_p^{\#} : C_{Y,f(p)} \to C_{X,p}$$

is a local homomorphism, where a local homomorphism is a ring homomorphism preserving the local ring structure. That is, if $\varphi : A \to B$ is a hom. with A, B local and m_A, m_B are the maximal ideals respectively, we have $\varphi^{-1}(m_B) = m_A$.
An isomorphism of Locally Ringed Spaces is a morphism with a two-sided inverse. Thus, a morphisn $(f, f^{\#})$ is an isomorphism if and only if f is a homeomorphism of the underlying topological space and $f^{\#}$ is an isomorphism of Sheaves.

Schemes! (Achievement Unlocked)

Definition

An *Affine Scheme* is a locally ringed space (X, O_X) which is isomorphic (as a locally ringed space) to the spectrum of a ring.

Definition

A *Scheme* is a Locally Ringed Space (X, O_X) in which every point p has an open neighbourhood U such that the topological space U together with the restricted Sheaf $O_{X|U}$ is an Affine Scheme.

We call X the underlying topological space and O_X its structure Sheaf. For notational convenience, we will write X to refer to the Scheme (X, O_X). A morphism of Schemes is a morphism of locally ringed spaces. An isomorphism is a morphism with a two-sided inverse. The topology on the Scheme is called the Zariski Topology.

Recovering a ring from an Affine Scheme

Given an affine scheme $X = (X, O_X)$ we can recover the ring from X. That is, we can find R such that Spec $R = X$, by taking the global sections. I.e. Consider $X = D(1)$ (Recall that $D(1)$ is the set of prime ideals on which the function $1 \in R$ does not vanish, that is $D(1) = \text{Spec } R$). Thus, we have

$$\Gamma(X, O_X) = \Gamma(\text{Spec } R, O_{\text{Spec } R}) = R$$

So if we want to attain the ring structure, we just look at the global sections.

Examples of Schemes

The examples we saw before of Spectra of rings are all examples of Affine Schemes. We present a couple of them here once again

Example

Let $R = \mathbb{R}[x]$. R is a UFD, one can show that the only prime ideals are $[0], [x - a]$ where $a \in$ and $x^2 + ax + b$ where $a, b \in \mathbb{R}$. Note that the nontrivial ideals are maximal, and thus when quotiented by, they yield a field. In particular, one can show that $\mathbb{R}/(x^2 + ax + b) \cong \mathbb{C}$. What does this look like? Well, we have the 0 point, all the real numbers and then for each irreducible quadratic we can interpret it as a pair of conjugate numbers (namely its roots over $\mathbb{C}$). So one can picture the spectrum as the Complex Plane folded over itself on over the real line.

Example Continued

In particular, we are "gluing" galois-conjugate points together. So for example $i, -i$ are glued together. Looking at functions on this space, let $f(x) = x^3 - 1$. Its value at $[x-2]$ is $7 \mod (x-2)$, similarly, its value at x^2+1 is

$$x^3 - 1 \equiv -x - 1 \mod (x^2+1)$$

Back To Manifolds

We construct a sheaf on $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ and let $O(U)$ be the C^∞-functions from $U \to \mathbb{R}$. The Stalk O_p at p consists of germs of functions near p and forms a ring. In particular, this is a local ring whose unique maximal ideal consists of those functions that vanish at p. $(\mathbb{R}^n, O)$ is a locally ringed space; one can check that the stalks are indeed local rings.

Definition

A *differentiable manifold* is a locally ringed space (M, O_M) where M is a second countable Hausdorff Space and O_M is a sheaf of rings such that (M, O_M) is locally isomorphic to $(\mathbb{R}^n, O)$.

One can think of a manifold as a Scheme modelled on $\mathbb{R}^n$ though we could easily change this definition to another locally ringed space such as $\mathbb{C}^n$.

Gluing Schemes

Let X_1, X_2 be schemes and let $U_1 \subseteq X_1, U_2 \subseteq X_2$ be open and let $\varphi : (U_1, O_{X_1}|_{U_1}) \to (U_2, O_{X_2}|_{U_2})$ be an isomorphism of locally ringed spaces. We can obtain a new scheme X by gluing U_1, U_2 together via φ. The resulting topological space is the standard topological gluing: the quotient of $X_1 \cup X_2$ by the relation $x_1 \sim \varphi(x_1)$ for $x_1 \in U_1$, and endowed with the quotient topology. This gives inclusion maps i_1, i_2 from $X_1, X_2 \to X$ and V is open in X if and only if $i_1^{-1}(V), i_2^{-1}(V)$ are open.

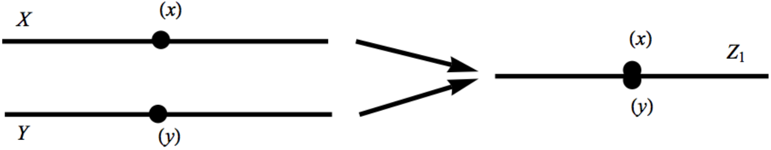
We can define a structure sheaf $\mathcal{O}_X$ as follows, for any open $V \subseteq X$:

$$\mathcal{O}_X(V) := \{(s_1, s_2) | s_1 \in \mathcal{O}_{X_1}(i_1^{-1}(V)) \text{ and } s_2 \in \mathcal{O}_{X_2}(i_2^{-1}(V)) \\ \text{and } \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2}\}$$

$\mathcal{O}_X$ is indeed a Sheaf, $(X, \mathcal{O}_X)$ is locally ringed and since X_1, X_2 are schemes, every point in X has a neighbourhood that is an affine scheme. Thus X is a Scheme.

Example: The Line with Two Origins

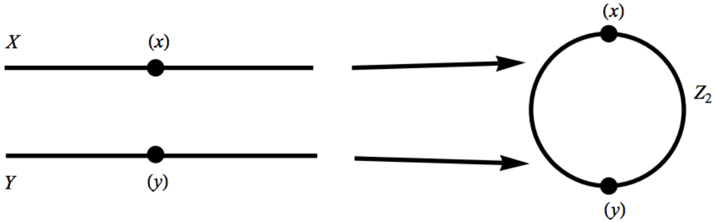
Let K be an algebraically closed field and let $X = \text{Spec } K[x]$ and $Y = \text{Spec } K[y]$ (i.e. the Affine line). Let $U = K[x, \frac{1}{x}]$, $V = K[y, \frac{1}{y}]$ (i.e. the line with the origin removed) and let $\phi : U \to V$ by $x \to y$ (effectively the identity map). Then, the resulting scheme obtained by gluing U and V can be visualized as



We get a line with the origin doubled. This is called the Line with Two Origins. This is a Scheme but is not an affine Scheme. It is also a classic example of a space which is not Hausdorff. In the language of Schemes, this is called a Nonseparated Scheme.

Projective Line

Let $X = \text{Spec } K[x]$, Let $U = K[x, \frac{1}{x}]$, $V = K[y, \frac{1}{y}]$ and let $\varphi : U \to X$ by $x \to \frac{1}{y}$. The resulting space can be visualized as



and is called the Projective Line.