AGS - INTRODUCTION TO SCHEMES

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Definition 0.1. (*Spectrum of a Ring*) Let R be a commutative ring. We define Spec R to be the set of prime ideals of R.

Definition 0.2. For an ideal I, we define $V(I)$ to be the set of prime ideals containing I.

Lemma 0.3. *Suppose I, J, K_i are ideals of R. Then:*

$$V(IJ) = V(I) \cup V(J)$$

$$V\Big(\sum_i K_i\Big) = \bigcap_i V(K_i)$$

$$V(I) \subseteq V(J) \text{ if and only if } \sqrt{I} \supseteq \sqrt{J}$$

Definition 0.4. (*Zariski Topology*) Based on the preceding lemma, we can define the *Zariski Topology* on Spec R to be the topology whose closed sets are $V(I)$ for some ideal $I \subseteq R$. Note that we will take the convention that R itself is not prime, so $V(R) = \phi$, and $V(\{0\}) = \text{Spec } R$.

Remark 0.5. An equivalent method of constructing the Zariski Topology is to view each element $r \in R$ as a function

$$r : \text{Spec } R \to \bigcup_{\mathfrak{p} \in \text{Spec } R} K(R/\mathfrak{p})$$

Where $r(\mathfrak{p}) = r_{\mathfrak{p}}$, its equivalence class in $R/\mathfrak{p}$, viewed as an element of the field of fractions. In this case, the Zariski closed sets are subsets of Spec R which are sent to a zero by sets of elements from R viewed as functions in this manner. It is clear that as in the classical case, we can extend to the ideal generated by the elements, and that $r(\mathfrak{p}) = 0$ if and only if $r \in \mathfrak{p}$, and we conclude that this definition agrees with the one given above.

Definition 0.6. We define the set of *regular functions* on Spec R to be R, with its elements viewed as functions on Spec R as in the preceding remark.

Definition 0.7. We recall that the localization of a commutative ring R at a prime ideal $\mathfrak{p}$ to be $(R\backslash\mathfrak{p})^{-1}R$, which we will denote $R_{\mathfrak{p}}$. We construct the sheaf $\mathcal{C}$ on the Zariski Topology such that for any open $U \subseteq \text{Spec } R$, $\mathcal{C}(U)$ is the set of functions

$$s : U \to \bigcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$$

Subject to the conditions that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ (s is a choice function on the set of $R_{\mathfrak{p}}$), and that for all $\mathfrak{p} \in U$, we can find an open neighbourhood $V \subseteq U$ and $a, b \in R$ such that for all $\mathfrak{q} \in V$, $b \notin \mathfrak{q} \in V$ and $s(q) = a/b$.

Remark 0.8. We obtain our restriction maps by simply restricting the domain of the functions in $\mathcal{C}(U)$. Moreover since the functions in $\mathcal{C}(U)$ locally may be viewed as an element of $K(R)$, if given an open cover of U, if a function returns zero on each open set of the cover, then it must locally zero at every point, and hence must be the zero function. Moreover we can "glue" elements of $\mathcal{C}(V_i)$ by taking the function $s(\mathfrak{p}) = s|_{V_i}(\mathfrak{p})$, $\mathfrak{p} \in V_i$, and this map is well defined by the compatibility condition.

Proposition 0.9. *Let $\mathfrak{p} \in$ Spec R. The stalk $\mathcal{C}_{\mathfrak{p}}$ is isomorphic to $R_{\mathfrak{p}}$*