Schemes on the horizon
Spectra of commutative rings

Michael L. Baker

University of Waterloo

mlbaker.org

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Overview

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Motivation

To any affine variety X we associated the coordinate ring $\Gamma(X) = A(X)$. $A(X)$ is always a rather special kind of ring: a finitely generated, reduced k-algebra. We now seek to "go the other way": to any commutative ring A, we wish to associate some kind of geometric object. These objects, which will generalise varieties, are called "affine schemes".

Recall that ideals of $A(X)$ correspond to ideals of A above $I(X)$. Under the hypothesis of algebraic closure, the Nullstellensatz tells us that maximal ideals of $A(X)$ correspond *exactly* to points of X. This suggests one can recover the variety as the set of maximal ideals of $A(X)$.

Motivation

It turns out to be more fruitful (for example, from a functorial standpoint) to consider the larger class of *prime* ideals.

To elaborate, suppose we had a ring homomorphism $f : A \to A'$. Then if we defined Spec A to be the set of all *maximal ideals* of A, we wouldn't be able to get a reasonable map Spec f : Spec $A' \to$ Spec A: your first guess is to take a maximal ideal $\mathcal{M} \in$ Spec A' and "pull it back" along f, that is, to define (Spec f)$(\mathcal{M}) = f^{-1}(\mathcal{M})$. However we may not always have $f^{-1}(\mathcal{M}) \in$ Spec A: it is in general *false* that preimages of maximal ideals (under ring homomorphisms) are maximal!

Our choice to consider *prime* ideals means we will have some "extra elements" which correspond not to points of X, but rather to subvarieties. These may seem unsettling, but they will turn out to be rather convenient.

The set Spec A

Let A be a commutative ring.

Definition

Denote by Spec A the set of all prime ideals $\mathcal{P} \subseteq A$.

We observe that each $f \in A$ gives rise to a "function" on Spec A: given $\mathcal{P} \in \operatorname{Spec} A$, we write $f(\mathcal{P})$ for the image of f under the natural map

$$A \to A/\mathcal{P} \to K(A/\mathcal{P})$$

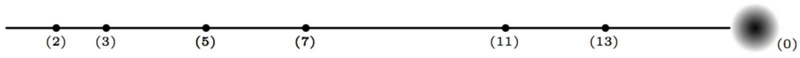
noting that $A/\mathcal{P}$ is an integral domain, so it makes sense to talk about its field of fractions $K(A/\mathcal{P})$.

Example

A good example to keep in mind is when $A = \Gamma(X) = A(X)$ is the coordinate ring of some affine variety X over an algebraically closed field. Then Spec A is basically just X (plus a few extra points corresponding to subvarieties), and each $f \in \Gamma(X)$ is viewed as a "function" on X.

Examples of Spec A

Example



Spec $\mathbb{Z}$

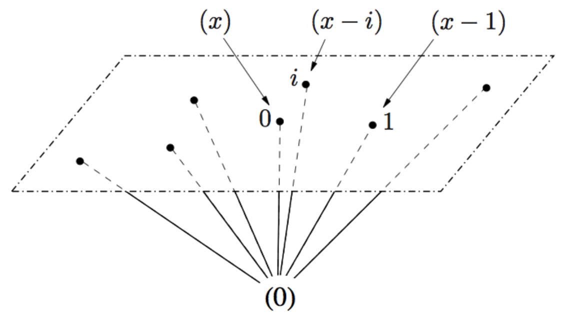
We have one point (p) for each prime $p \in \mathbb{Z}$ in addition to an extra point (0), usually called the *generic point*.

It is common to scale the points $\mathcal{P}$ of Spec A according to the size of the *residue field*, $\kappa(\mathcal{P}) = K(A/\mathcal{P})$. You may also see a point $\mathcal{P}$ drawn as "smudged", as above, to suggest that (*geometrically* speaking!) it contains other points in the diagram.

We now give some examples of Spec A for $A = \mathbb{Z}$, $\mathbb{C}[x]$, $\mathbb{Z}[x]$. Another interesting example is the ring of integers of an algebraic number field.

Examples of Spec A

Example

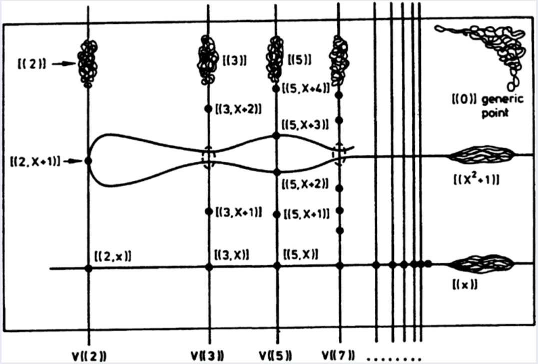


Spec $\mathbb{C}[x]$

$\mathbb{C}$ is algebraically closed. Also, $k[x]$ is a PID when k is a field, so all nonzero primes are maximal: its Krull dimension is 1. Thus Spec $\mathbb{C}[x]$ is just a "complex line" of maximal ideals sitting above the prime (0).

Examples of Spec A

Example



Mumford's drawing of Spec $\mathbb{Z}[x]$, the **arithmetic surface**

The Krull dimension of $\mathbb{Z}[x]$ is 2.

Examples of Spec A

Example ("polynomial remainder theorem" from high school)

Let $A = \mathbb{C}[x]$ and $p \in \mathbb{C}[x]$. If $\alpha \in \mathbb{C}$, then $(x - \alpha)$ is prime in $\mathbb{C}[x]$, and one can identify $\kappa((x - \alpha))$ with $\mathbb{C}$ naturally, so that the value of p at the point $(x - \alpha) \in \operatorname{Spec} \mathbb{C}[x]$ is the number $p(\alpha)$.

Example

More generally, if A is the coordinate ring of an affine variety X over an algebraically closed field k, and $\mathcal{P}$ is the maximal ideal corresponding to $x \in X$, then $\kappa(\mathcal{P}) = k$ and $f(\mathcal{P})$ is the value of f at x in the usual sense.

Note that $f \in A$ takes values in fields that *vary* from point to point. Moreover, f is not determined by its values: take the "$\mathbb{C}$-dual numbers" $A = \mathbb{C}[x]/(x^2)$ and take $f = x \in A$. Then $0 \neq f$ vanishes on all of Spec A.

Definition

Elements of A are called **regular functions** on Spec A.

Topology on Spec A

An element $f \in A$ such that $f \in \mathcal{P}$ corresponds to a "function" that is zero on $\mathcal{P}$. We now use regular functions to define a topology on Spec A. Note the obvious analogue with the "zero sets" of the classical case.

The idea is that we want to make each $f \in A$ behave like a continuous function (even though this doesn't make sense). The fields $K(A/\mathcal{P})$ certainly all contain a 0; if f is to be "continuous" then $f^{-1}(\{0\})$ should definitely be a closed set.

Definition

For each $S \subset A$, we define

$$\begin{aligned} V(S) &= \{x \in \text{Spec } A : f(x) = 0 \text{ for all } f \in S\} \\ &= \{\mathcal{P} \in \text{Spec } A : \mathcal{P} \supset S\}. \end{aligned}$$

Of course, $V(S) = V(\mathcal{I})$ if $\mathcal{I}$ is the ideal generated by S in A.

Topology on Spec A

The following proposition from Hartshorne assures us that the sets of the form $V(\mathcal{I})$ do indeed constitute the closed sets of a topology on Spec A.

Proposition

(a) $V(\mathcal{I}\mathcal{J}) = V(\mathcal{I}) \cup V(\mathcal{J})$ (closure under finite unions).

(b) $V(\sum \mathcal{I}_\alpha) = \bigcap V(\mathcal{I}_\alpha)$ (closure under arbitrary intersections).

(c) $V(\mathcal{I}) \subseteq V(\mathcal{J})$ if and only if $\sqrt{\mathcal{I}} \supseteq \sqrt{\mathcal{J}}$.

Proof

(a) If $\mathcal{P} \supseteq \mathcal{I}$ or $\mathcal{P} \supseteq \mathcal{J}$, then clearly $\mathcal{P} \supseteq \mathcal{I}\mathcal{J}$. Conversely if $\mathcal{P} \supseteq \mathcal{I}\mathcal{J}$ but $\mathcal{P} \not\supseteq \mathcal{J}$, then $\exists b \in \mathcal{J}$ with $b \notin \mathcal{P}$. Now for any $a \in \mathcal{I}$ we have that $ab \in \mathcal{P}$ and therefore $a \in \mathcal{P}$ by primality. We conclude $\mathcal{P} \supseteq \mathcal{I}$.

(b) $\mathcal{P} \supseteq \sum \mathcal{I}_\alpha$ if and only if $\mathcal{P} \supseteq \mathcal{I}_\alpha$ for all α, simply because $\sum \mathcal{I}_\alpha$ is the smallest ideal containing all the ideals $\mathcal{I}_\alpha$.

(c) Note that $\sqrt{\mathcal{I}}$ is the intersection of all prime ideals containing $\mathcal{I}$.

Sheaf of rings on Spec A

The final piece of data carried by Spec A is its sheaf of rings ("structure sheaf"), which we now define. Recall $A_{\mathcal{P}}$ denotes the localisation of A at $\mathcal{P}$, that is, $(A \setminus \mathcal{P})^{-1}A$.

Definition

For each open $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathcal{P} \in U} A_{\mathcal{P}}$ such that $s(\mathcal{P}) \in A_{\mathcal{P}}$ for each $\mathcal{P}$, and s is "locally a quotient of elements of A", that is, for each $\mathcal{P} \in U$, there is a neighbourhood V of $\mathcal{P}$, with $V \subseteq U$, and $a, f \in A$ such that for each $\mathcal{Q} \in V$, $f \notin \mathcal{Q}$, and $s(\mathcal{Q}) = a/f$ in $A_{\mathcal{Q}}$.

This is similar to the definition of regular functions on a variety, except instead of considering functions to a field, we look at functions into the various local rings.

Sheaf of rings on Spec A

Note that $\mathcal{O}(U)$ is a commutative ring with 1. With the natural restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$, $\mathcal{O}$ becomes a sheaf on Spec A.

Definition

$\mathcal{O}$ is called the **structure sheaf** (or the **sheaf of regular functions**) on $X = \text{Spec } A$.

With this definition, the relationship between Spec A and A generalises that between an affine variety and its coordinate ring. In particular, we will see that the ring $\Gamma(\text{Spec } A, \mathcal{O})$ of global sections of $\mathcal{O}$ is precisely A.

The spectrum of A

Definition

The **spectrum** of A is the pair $(\operatorname{Spec} A, \mathcal{O})$.

The open sets in the topology on $X = \operatorname{Spec} A$ are just the complements of sets $V(S)$. The ones corresponding to *singletons* S will play a special role: they form a *base* for the topology, and they are again spectra of rings.

Definition

For any $f \in A$, let $X_f = D(f)$ be the complement of $V(f)$. This is called the **distinguished** (or **basic**) **open subset associated with** f.

The points of X_f (prime ideals of A not containing f) are in bijection with the prime ideals of A_f via sending $\mathcal{P} \subset A$ to $\mathcal{P}A_f \subset A_f$. Hence we identify X_f with the points of $\operatorname{Spec} A_f$.

An "affine scheme" will be defined as something isomorphic to a spectrum.

Sheaf of rings on Spec A

The following proposition gives us some information about the structure sheaf $\mathcal{O}$ on Spec A.

Proposition (2.2 in Hartshorne)

Let A be a ring, and let (Spec $A, \mathcal{O}$) be its spectrum.

(a) For any $\mathcal{P} \in$ Spec A, the stalk $\mathcal{O}_{\mathcal{P}}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_{\mathcal{P}}$.

(b) For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localisation A_f.

(c) In particular, Γ(Spec $A, \mathcal{O}) \cong A$.

Proof

Long. See Hartshorne, p. 71.

Ringed spaces

To each commutative ring A, we have now associated its spectrum $(\text{Spec } A, \mathcal{O})$. We want to get a functor out of this, so we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is that of a *locally ringed space*, which will be treated next time. We conclude with the following preliminary.

Recall from last week the concept of a *direct image sheaf* $f_* \mathcal{F}$.

Definition

A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A **morphism** of ringed spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair $(f, f^\#)$ consisting of a continuous map $f : X \to Y$ and a morphism $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.