Algebraic aspects of sheaves

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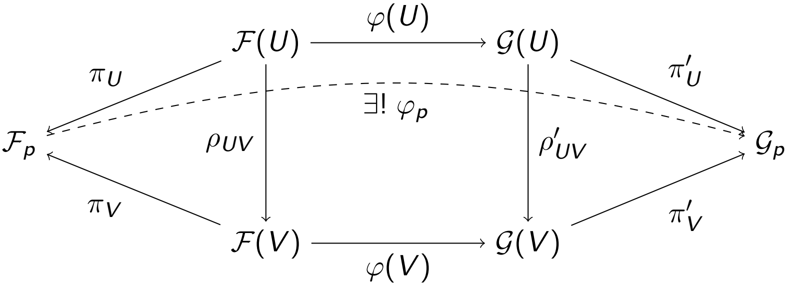
Overview

Induced maps on stalks

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves (of abelian groups) on X, and let $p \in X$. By considering the following diagram



it is clear, by the "initiality" of $(\mathcal{F}_p, \pi)$ among all cones from the direct system $(\mathcal{F}(U), \rho)_{U \ni p \text{ open}}$, that we obtain a map $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$, which is a morphism of abelian groups (*why* do those composites form a cone to $\mathcal{G}_p$?)

Induced maps on stalks

The definition of $\mathcal{F}_p := \varinjlim \mathcal{F}(U)$ says that this map $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is the unique such map which, for all open sets $U \subseteq X$, causes the diagram below to commute:

$$\begin{array}{ccc} \mathcal{F}_p & \overset{\varphi_p}{\dashrightarrow} & \mathcal{G}_p \\ {\scriptstyle \pi_U}\uparrow & & \uparrow{\scriptstyle \pi'_U} \\ \mathcal{F}(U) & \underset{\varphi(U)}{\longrightarrow} & \mathcal{G}(U) \end{array} \qquad (*)$$

That is, taking the germ and then following φ_p is the same as following $\varphi(U)$ and then taking the germ: in symbols, $\varphi_p(s_p) = \varphi(s)_p$ (we write $\varphi(s)$ rather than $\varphi(U)(s)$ – *why is this sensible*?).

Definition

$\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is called the **induced map** (by φ) **on the stalk at** p.

Working at the level of stalks

Now suppose $\mathcal{F}$ and $\mathcal{G}$ are in fact *sheaves*. Often, when trying to prove some property of such a morphism φ, it is conveniently sufficient to "work at the level of stalks". Of course, one must be careful. We now see an example of this.

Proposition

φ is an isomorphism if and only if the induced map on the stalk $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for each $p \in X$.

We now present the proof of this proposition, which is enlightening because it yields insight into why the "local behaviour" of sheaves is a desirable thing to have. **The above is in general false for presheaves!**

Pay meticulous attention to which hypotheses are used in which parts of the proof. You may notice an enigmatic creature crouching in the shadows of the last part of the proof. It is silently warning us of an oddity to come.

Proof of Proposition

($\to$) [Note: Hartshorne writes this off as "clear", so I proved it]. Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is iso. Then φ admits an inverse morphism, $\varphi^{-1} : \mathcal{G} \to \mathcal{F}$. Let $p \in X$. For any open $U \subseteq X$, smash together two of (*) appropriately:

$$\begin{array}{ccccc}
\mathcal{F}_p & \xrightarrow{\quad \varphi_p \quad} & \mathcal{G}_p & \xrightarrow{\quad (\varphi^{-1})_p \quad} & \mathcal{F}_p \\
{\scriptstyle \pi_U}\uparrow & & \uparrow{\scriptstyle \pi'_U} & & \uparrow{\scriptstyle \pi_U} \\
\mathcal{F}(U) & \xrightarrow[\quad \varphi(U) \quad]{} & \mathcal{G}(U) & \xrightarrow[(\varphi^{-1})(U) = \varphi(U)^{-1}]{} & \mathcal{F}(U)
\end{array}$$

The composite of the bottom two arrows is just $\mathrm{id}_{\mathcal{F}(U)}$. Since the two individual squares commute, the outside square does too (why?). Thus, $(\varphi^{-1})_p \circ \varphi_p = \mathrm{id}_{\mathcal{F}_p}$ (because $\mathrm{id}_{\mathcal{F}_p}$ *works*, and such a map is unique by considering the squares (*) for the identity morphism $\mathrm{id} : \mathcal{F} \to \mathcal{F}$). Similarly, $\varphi_p \circ (\varphi^{-1})_p = \mathrm{id}_{\mathcal{G}_p}$ (smash it on the other side instead)! Thus, $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is iso, as desired.

Proof of Proposition

($\leftarrow$) Suppose now that each $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is iso. If we can show that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is iso for each open $U \subseteq X$, then we may define an inverse morphism $\psi : \mathcal{G} \to \mathcal{F}$ simply by $\psi(U) = \varphi(U)^{-1} : \mathcal{G}(U) \to \mathcal{F}(U)$ (why is ψ a morphism/natural transformation?). Thus, we prove each $\varphi(U)$ is bijective.

Injectivity: suppose $s \in \mathcal{F}(U)$ is such that $\varphi(s) = 0 \in \mathcal{G}(U)$. We claim $s = 0$. Indeed, for any point $p \in U$, the germ $\varphi(s)_p = \varphi_p(s_p) = 0 \in \mathcal{G}_p$. Now, φ_p is iso, hence injective, implying $s_p = 0 \in \mathcal{F}_p$. So for each $p \in U$, there is a neighbourhood V_p of p, with $V_p \subseteq U$, such that $s|_{V_p} = 0$. Since $\{V_p\}_{p \in U}$ covers U, the identity axiom implies $s = 0 \in \mathcal{F}(U)$, as desired. So $\varphi(U)$ is injective.

Proof of Proposition

Surjectivity: let $t \in \mathcal{G}(U)$. For all $p \in U$, consider the germ $t_p \in \mathcal{G}_p$. Since $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is iso, it is surjective, so there is $s_p \in \mathcal{F}_p$ with $\varphi_p(s_p) = t_p$. Now we build an element $s \in \mathcal{F}(U)$. For each $p \in U$, let s_p be represented on a neighbourhood V_p of p, by some $s(p) \in \mathcal{F}(V_p)$, choosing V_p small enough so that $\varphi(s(p)) = t|_{V_p}$; this is possible since $\varphi(s(p))$ and t have the same germ at p. Note that $s(p)|_{V_p \cap V_q}$ and $s(q)|_{V_p \cap V_q}$ are two sections of $\mathcal{F}(V_p \cap V_q)$ which are both sent by φ to $t|_{V_p \cap V_q}$ (why?). Hence, by injectivity of φ proved above (!), they are equal.

By the gluing axiom of $\mathcal{F}$, there is $s \in \mathcal{F}(U)$ such that $s|_{V_p} = s(p)$ for all $p \in U$. We claim $\varphi(s) = t$. Indeed, $\varphi(s)$ and t are two sections of $\mathcal{G}(U)$ and for each $p \in U$, we have $\varphi(s)|_{V_p} = t|_{V_p}$ (as we arranged). Hence by the identity axiom of $\mathcal{G}$, we get $\varphi(s) = t$. Thus, $\varphi(U)$ is surjective.

This completes the proof that $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism.

Kernels, cokernels, and images

For many algebraic structures, one can define the useful notions of kernel, cokernel, and image (indeed, all three of these things have abstract categorical definitions). We now define analogs of these for presheaves.

Definition

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. The presheaf given by

- $U \mapsto \ker(\varphi(U))$ is called the **presheaf kernel** of φ
- $U \mapsto \operatorname{coker}(\varphi(U))$ is called the **presheaf cokernel** of φ
- $U \mapsto \operatorname{im}(\varphi(U))$ is called the **presheaf image** of φ.

Hiccup

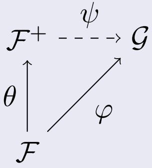
If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*, then the presheaf kernel is a sheaf, but the other two need not be! Hence, we now discuss how we can "turn a presheaf into a sheaf" in the most general way.

Sheafification: a process of refinement

Let $\mathcal{F}$ be a presheaf on X. Then there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ through which *any morphism*, from $\mathcal{F}$ to a sheaf, factors.

Proposition (Universal Property of Sheafification)

Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for *any* sheaf $\mathcal{G}$ and morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a *unique* morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes:



Furthermore, $\mathcal{F}^+$ is unique up to (unique!) isomorphism. $\mathcal{F}^+$ is called the **sheafification** of $\mathcal{F}$ (clearly awesome terminology).

Teach Me How To Sheafify (Explicit Version)

So how does one actually *construct* the sheafification? The idea is to define $\mathcal{F}^+(U)$ to be the set of all "well-behaved sections", that is, functions $s : U \to \coprod_{p \in U} \mathcal{F}_p$ such that:

- $f(p) \in \mathcal{F}_p$ for all $p \in U$
- for all $p \in U$, there exists a neighbourhood V_p of p, with $V_p \subseteq U$, and some $t \in \mathcal{F}(V_p)$, such that for all $q \in V_p$, $t_q = s(q)$.

Here, of course, t_q denotes the germ of t at q. The morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ to use is clear: merely send each section $s \in \mathcal{F}(U)$ to its "bundle of germs", $(s_p)_{p \in U}$. (Reminds you of $V \cong V^{**}$, doesn't it?)

The first condition merely says that s tells us "local data" about itself at each point $p \in U$. So if $\mathcal{F}$ is already a sheaf, this is (by the gluing axiom) already enough to actually reconstruct an element of $\mathcal{F}(U)$. We impose the second condition to ensure that we *only consider* those "bundles of local data" that are well-behaved enough to be lifted.

Kernels, cokernels, and images (for sheaves)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of *sheaves* on X.

Having discussed sheafification, we can finally return to our original problem of defining the kernel, cokernel, and image of φ. But first, we make the following definition.

Definition

A **subsheaf** of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of $\mathcal{F}'$ are induced by those of $\mathcal{F}$.

Note for any point $p \in X$, the stalk $\mathcal{F}'_p$ is a subgroup of $\mathcal{F}_p$ (why?).

Definition

The **kernel** of φ, denoted $\ker \varphi$, is defined to be the presheaf kernel of φ.

Verify that $\ker \varphi$ is a sheaf, indeed a subsheaf of $\mathcal{F}$.

Kernels, cokernels, and images (for sheaves)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of *sheaves* on X.

Definition

φ is called **injective** if $\ker \varphi = 0$. Thus φ is injective if and only if $\varphi(U)$ is injective for all open $U \subseteq X$.

Definition

The **image** of φ, denoted $\operatorname{im} \varphi$, is defined to be the sheafification of the presheaf image of φ.

By the universal property, there is a natural map $\operatorname{im} \varphi \to \mathcal{G}$ which turns out to be injective (exercise 1.4), so $\operatorname{im} \varphi$ can be identified with a subsheaf of $\mathcal{G}$.

Definition

If $\operatorname{im} \varphi = \mathcal{G}$, we call φ **surjective**.

Kernels, cokernels, and images (for sheaves)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of *sheaves* on X.

Definition

A sequence

$$\ldots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \to \ldots$$

of sheaves and morphisms is **exact** if at each stage $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$.

Thus, $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact iff φ is injective, and $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact iff φ is surjective.

Definition

If $\mathcal{F}'$ is a subsheaf of the sheaf $\mathcal{F}$, we define the **quotient sheaf** $\mathcal{F}/\mathcal{F}'$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

Check that for any point p, the stalk $(\mathcal{F}/\mathcal{F}')_p$ is the quotient $\mathcal{F}_p/\mathcal{F}'_p$.

Kernels, cokernels, and images (for sheaves)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of *sheaves* on X.

Definition

The **cokernel** of φ, denoted $\operatorname{coker} \varphi$, is defined to be the sheafification of the presheaf cokernel of φ.

Warning

We saw that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subseteq X$. However, be careful: if φ is surjective then the maps $\varphi(U)$ need not be!

On the other hand, a sequence of sheaves and morphisms is exact if and only if the corresponding sequence of stalks and induced maps is exact for each $p \in X$. In particular, φ is surjective if and only if φ_p is surjective for each $p \in X$.

Operations on sheaves

Let $f : X \to Y$ be a continuous map of topological spaces. Until now we have only concerned ourselves with morphisms between sheaves on the same space X. We now look at a way of transporting a sheaf on one space to a sheaf on another, via f. We will use this soon when we start schemes.

Definition

We define the **direct image sheaf** $f_*\mathcal{F}$ on Y by putting, for open $V \subseteq Y$, $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$; note $f^{-1}(V)$ is open in X by continuity of f.

Definition

We define the **inverse image sheaf** $f^{-1}\mathcal{G}$ on X to be the sheafification of the presheaf given by $U \mapsto \varinjlim \mathcal{G}(V)$, the limit being taken over all open $V \supseteq f(U)$.

Note that f_* is a functor from (the category of) sheaves on X to sheaves on Y. Similarly, f^{-1} is a functor from sheaves on Y to sheaves on X.