Grothendieck Agriculture: The method of stalking

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Overview

Presheaves

For the rest of the talk, let X be a topological space.

Definition

A **presheaf** $\mathcal{F}$ (of abelian groups, say) **on** X consists of the data:

- for each open $U \subseteq X$, an abelian group $\mathcal{F}(U)$
- for each inclusion $V \subseteq U$ of open sets of X, a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ of abelian groups, called **restriction maps**

such that

- $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map on $\mathcal{F}(U)$
- if $W \subseteq V \subseteq U$ are open, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Elements $s \in \mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U. We often write $s|_V$ rather than $\rho_{UV}(s) \in \mathcal{F}(V)$, and call this the **restriction** of s to V.

Sheaves

Definition

A presheaf $\mathcal{F}$ on X is called a **sheaf** if, for all open $U \subseteq X$ and any open cover $\{V_i\}$ of U, it further satisfies

- [IDENTITY] if $s, s' \in \mathcal{F}(U)$ agree on each V_i then in fact $s = s'$
- [GLUING] given $s_i \in \mathcal{F}(V_i)$ for all i that agree on the overlaps, there exists $s \in \mathcal{F}(U)$ whose restriction to each V_i is precisely s_i (note that by IDENTITY above, s is unique!)

Remark

Sheaves can also be defined as topological spaces "over" X satisfying certain properties. This is the notion of an **étalé space**; we will probably come back to it later.

Morphisms

Definition

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A **morphism of presheaves**

$$\varphi : \mathcal{F} \to \mathcal{G}$$

consists of, for each open $U \subseteq X$, a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ of abelian groups which "commutes with restriction" in the sense that whenever $V \subseteq U$ is an inclusion of open sets, we have $\varphi(f)|_V = \varphi(f|_V)$, or to be more precise, $\rho'_{UV} \circ \varphi(U) = \varphi(V) \circ \rho_{UV}$ where ρ and ρ' are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$ respectively (draw the commutative diagram)!

We use the same definition for a **morphism of sheaves** (this is fine since every sheaf is a presheaf).

Stalks

We often want to understand the "local behaviour" of a sheaf, at a particular point. This is facilitated by the following definition.

Definition

Given a sheaf $\mathcal{F}$ on X, and $p \in X$, we equip the set

$$\{(f, U) : U \subseteq X \text{ open}, \; p \in U, \; f \in \mathcal{F}(U)\}$$

with the equivalence relation $\sim$ defined by declaring $(f, U) \sim (g, V)$ iff f and g agree on some open set $W \subseteq U \cap V$ with $p \in W$.

The equivalence classes under $\sim$ are called **germs of $\mathcal{F}$ at** p. The set of all germs is denoted $\mathcal{F}_p$ and is called the **stalk** of $\mathcal{F}$ at p. It is an abelian group in the obvious way.

We will see that in many cases, to verify that a morphism of sheaves possesses a given property, one can just check it on stalks.

Categories: a new language

In mathematics, we commonly look at a particular type of mathematical creature, establish some notion of "structure-preserving transformation" between them, and then study how the objects interact via those maps. Loosely speaking, the notion of a *category* will capture this "class of objects together with distinguished maps between them" concept.

For example, we can consider sets (and functions), vector spaces (and linear maps), groups (and group homomorphisms), and as you might have guessed, even sheaves on X (and morphisms of sheaves)! These will all form categories.

Category theory is a language providing concepts that subsume and make obvious the similarities between entire families of analogous constructions, ubiquitous throughout mathematics. Category theory is to mathematics as a "design patterns" book is to programming.

Categories

Definition

A **category** $\mathcal{C}$ consists of

- a class $\operatorname{Ob}\mathcal{C}$ of **objects**
- for each ordered pair (A, B) of objects, a class $\operatorname{Hom}(A, B)$ of **morphisms** (or **arrows**, **maps**, etc.), often represented as $f : A \to B$
- for each ordered triple (A, B, C) of objects, a **composition rule**

$$\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$$

subject to the following:

- each object A of $\mathcal{C}$ admits an **identity morphism** $\mathrm{id}_A : A \to A$ such that for all $f \in \operatorname{Hom}(A, B)$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$
- composition is **associative**, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$ whenever f, g, h are such that both sides of the equation are defined.

Examples of categories

Here are some examples of categories (this is but a tiny glimpse into just how pervasive they are):

- groups (with group homomorphisms), abelian groups (with group homomorphisms), rings (with ring homomorphisms), commutative rings (with ring homomorphisms)
- vector spaces (with linear maps), R-modules (with R-module homomorphisms)
- graphs (with graph homomorphisms)
- smooth manifolds (with smooth maps), differentiable manifolds (with differentiable maps), complex manifolds (with holomorphic maps)
- topological spaces (with continuous maps), metric spaces, Banach spaces, Hilbert spaces
- categories (with *functors*), functors between two fixed categories (with *natural transformations*)
- partially ordered sets, or even preorders ($\exists$ morphism $a \to b$ iff $a \leq b$)

Functors

You may have noticed that mathematicians like to take objects and associate other objects to them. Often, this is done in such a way that morphisms between the original objects give rise to morphisms between the new objects in a nice fashion.

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the data:

- for each object A of $\mathcal{C}$, an object $F(A)$ (or just FA) of $\mathcal{D}$
- for each morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$

such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$. We can also define **contravariant functors** which "flip the morphisms", i.e. send $f : A \to B$ to $F(f) : F(B) \to F(A)$, subject to analogous conditions (write them out)!

Examples of functors

Functors, like categories (and just about every other categorical concept) crop up *everywhere*; just keep your eyes peeled. Some examples:

- The dual space V^* (and double dual V^{**}) of a vector space V
- The general linear group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over a field F (or even a ring)
- The group of units $U(R)$ of a ring R
- The commutator $[G, G]$ and the abelianisation $G/[G, G]$ of a group G
- So-called **forgetful functors** that send a group/ring/etc. to their underlying set
- Homotopy, homology, and cohomology in algebraic topology
- The Lie algebra $\mathfrak{g}$ of a Lie group G
- The tangent space T_pM of a smooth manifold M at a point p.

Natural transformations

Finally we come to *transformations between functors*. This concept lends full rigour to the familiar terms "natural" and "canonical" (which you may have previously only understood heuristically). Roughly speaking, natural transformations are to functors as homotopies are to paths.

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be (covariant!) functors. A **natural transformation**

$$\eta : F \to G$$

consists of, for each object A of $\mathcal{C}$, a morphism $\eta_A : F(A) \to G(A)$ of $\mathcal{D}$, called the **component of** η **at** A, such that for any morphism $f : A \to B$ in $\mathcal{C}$,

$$\eta_B \circ F(f) = G(f) \circ \eta_A.$$

In other words, the diagram on the next page commutes.

Natural transformations

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

(the **naturality square** of η for f)

Definition

A morphism $f : A \to B$ of $\mathcal{C}$ is called an **isomorphism** (or **iso**, or **invertible**) if there exists a morphism $f^{-1} : B \to A$ of $\mathcal{C}$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Note f^{-1} must be a morphism of $\mathcal{C}$!

If all the components η_A of a natural transformation $\eta : F \to G$ are iso, we say η is a **natural isomorphism**. In this case, the maps η_A^{-1} are the components of an "inverse" natural transformation, $\eta^{-1} : G \to F$.

Directed sets

Since we seek another, more abstract interpretation of the stalk of a sheaf, we now discuss direct limits. These are a special case of an extremely general notion from category theory, which is (confusingly) called a *colimit*.

We will first need the following definition (which incidentally is also used to define nets in topological spaces à la PMATH 753).

Definition

A **directed set** is a nonempty set I equipped with a reflexive, transitive relation $\leq$ (i.e. a **preorder**) such that for all $\nu, \nu' \in I$ there exists $\mu \in I$ with $\nu \leq \mu$ and $\nu' \leq \mu$.

That is, every pair of elements has an upper bound.

Direct systems

Definition

Let $(I, \leq)$ be a directed set, and $\mathcal{C}$ be a category. A **direct system in $\mathcal{C}$ over I** consists of

- a collection $\{A_i\}_{i \in I}$ of objects of $\mathcal{C}$ indexed by I
- for all $i \leq j$, a morphism $f_{ij} : A_i \to A_j$ of $\mathcal{C}$

such that

- f_{ii} is the identity map on A_i
- $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

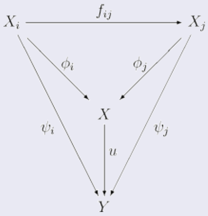
More tersely, a direct system over I is a functor from I to $\mathcal{C}$ where we view I as a category in the obvious way. Such functors may be called **diagrams**.

In the context of defining stalks, we will take $(I, \leq)$ to be the collection of open neighbourhoods of $p \in X$, ordered by *containment* $\supseteq$ (not inclusion)!

Direct limits

Definition

Suppose (X_i, f_{ij}) is a direct system in $\mathcal{C}$. The **direct limit** of (X_i, f_{ij}) is an object X of $\mathcal{C}$ together with morphisms $\phi_i : X_i \to X$ such that $\phi_i = \phi_j \circ f_{ij}$ for all $i \leq j$. This pair (X, ϕ_i) must be "universal" (better, "initial") in the sense that for any other such pair (Y, ψ_i) there exists a unique morphism $u : X \to Y$ making the diagram



commute for all $i \leq j$. We often denote the direct limit as $X = \varinjlim X_i$, with the direct system (X_i, f_{ij}) being understood.

Existence of direct limits

Warning

If $\mathcal{C}$ is an arbitrary category, direct limits need not exist!

On the other hand, if the category $\mathcal{C}$ is one of algebraic structures, direct limits will almost always exist: indeed we can concretely describe the direct limit as a quotient (= set of equivalence classes) of a disjoint union:

$$\varinjlim X_i = \bigsqcup_i X_i \Big/ \sim$$

where if $x_i \in X_i$ and $x_j \in X_j$, we declare $x_i \sim x_j$ iff for some $k \in I$ we have $f_{ik}(x_i) = f_{jk}(x_j)$. This is the *exact same idea* behind our first (concrete) definition of the stalk: two elements of the disjoint union are considered the same iff they "eventually become equal" in the direct system.

The category $\mathfrak{Top}(X)$ associated to X

Now that we have discussed the fundamental concepts of category theory, we can entertain ourselves by recasting the basic notions of sheaf theory in this new, elegant language. The following definition plays a crucial role.

Definition

Given a topological space X, we define a category $\mathfrak{Top}(X)$ by declaring the objects to be precisely the open subsets $U \subseteq X$, and the morphisms to be precisely the inclusion maps.

Thus, $\mathrm{Hom}(V, U)$ consists of one element if $V \subseteq U$; otherwise it is empty.

Note that $\mathfrak{Top}(X)$ is a *special case* of a "poset category" (see the list of examples): it is the category corresponding to the topology of X (i.e. the collection of its open sets), viewed as a partially ordered set under the inclusion relation $\subseteq$.

Recasting sheaf theory into the new language

Observation

- A presheaf (of abelian groups) on X is simply a *contravariant functor* $\mathfrak{Top}(X) \to \mathbf{Ab}$ (where **Ab** is the category of abelian groups and group homomorphisms).
- One can recast sheaves in this more succinct language, but it is slightly more technical (makes use of the concept of an *equalizer*). Feel free to look it up.
- As you may have guessed, a morphism of presheaves is nothing more than a *natural transformation* between them (since, as mentioned above, presheaves are functors).
- If $\mathcal{F}$ is a sheaf on X, and $p \in X$, then the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at p is just the *direct limit*

  $$\varinjlim \mathcal{F}(U)$$

  as U ranges over all open sets containing p.