Algebraic Geometry III: Sheaves and Things

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In Geometry one thing we care about is given our structure, what does the structure behave locally? In the context of Differential Geometry, to each point we associate a vector space, namely the Tangent space at that point. Similarly, given a topological space X we want to define additional structure on top of X that will capture what is going on locally. Before actually defining what a Sheaf is, I want to provide a motivating example. It might help to think of this if you ever find yourself getting lost in a sea of abstractness.

EXAMPLE **0.1.** Let $X = \mathbb{R}^n$ with the Euclidean topology. On each open set U, look at the abelian of group (or if one wants the ring) of differentiable functions on U. Lets denote this $\mathcal{F}(U)$. Given a differentiable function on an open set, we can restrict it to a smaller open set, obtaining another differentiable function on the smaller space. That is, if $V \subset U$ open, then we have a restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$.

Next, take a differentiable function on a big open set U, restrict it to a smaller open set V and now further restrict to a smaller set W. The result is the same as if one just restricted U down to W. That is to say that the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & & \mathcal{F}(V) \arrow[dl, "\rho_{VW}"] \\
& \mathcal{F}(W) &
\end{tikzcd}$$

commutes, where $\mathcal{F}(U)$ denotes the space of differentiable functions from $U \to \mathbb{R}$ and ρ_{UV} is the restriction map.

Next, suppose we have two differentiable functions f, g on some open set U and suppose that $\{V_i\}_{i \in I}$ is a cover of U by open sets. Well if $f = g$ on each V_i then since the V_i cover U, $f = g$ on U.

Finally, suppose we have an open cover $\{V_i\}_{i \in I}$ of U and we have a collection of differentiable functions $f_i \in \mathcal{F}(V_i)$ such that if $V_i \cap V_j \neq \emptyset$ then $f_i = f_j$ on $V_i \cap V_j$. Then we can construct a new differentiable function on U by gluing these functions together. In other words, there exists $f \in \mathcal{F}(U)$ such that $\rho_{U,V_i}(f) = f_i$ for all $i \in I$.

DEFINITION **0.2.** [PRESHEAF OF ABELIAN GROUPS] Let X be a topological space. A **presheaf** $\mathcal{F}$ of abelian groups on X consists of the data

- For every open subset $U \subseteq X$, have an associated abelian group $\mathcal{F}(U)$
- If $V \subseteq U$ are open sets, have a group homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (i.e. a restriction map)

subject to the conditions

1. $\mathcal{F}(\emptyset) = 0$
2. ρ_{UU} is the identity map
3. If $W \subseteq V \subseteq U$ are three open subsets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

An element $s \in \mathcal{F}(U)$ is called a section of F over U. We let $s|_V$ denote $\rho_{UV}(s) \in \mathcal{F}(V)$ and we call it the restriction of s to V. One can define a presheaf of rings, sets in the obvious analogous way to abelian groups.

REMARK **0.3.** Given a topological space X, we consider the category of open sets where the objects are open sets and the morphisms are inclusion. One can check that the data given by the Presheaf is precisely the data of a contravariant functor (Whatever that means) from the category of open sets of X to the category of sets.

DEFINITION **0.4.** [SHEAF] A Presheaf $\mathcal{F}$ on a topological space is said to be a Sheaf if it satisfies the following supplementary conditions

1. (Identity) If U is an open set, if $\{V_i\}$ is an open covering of U and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i then $s = 0$. An alternate way of thinking about this is we have two elements u, v such that $u|_{V_i} = v|_{V_i}$ on each V_i then $u = v$.
2. (Gluing) If U is an open set, if $\{V_i\}$ is an open covering of U and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each i, j $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. Note that the identity condition gives us that this s is unique.

EXAMPLE **0.5.** [RESTRICTION SHEAF] Let $\mathcal{F}$ be a sheaf on X and let $U \subset X$ be open. Define the restriction of $\mathcal{F}$ to U ,denoted by $\mathcal{F}|_U$, by putting $(\mathcal{F}|_U)(V) = \mathcal{F}(V)$ for all open subsets $V \subset U$.

EXAMPLE **0.6.** [MORE APPLIED EXAMPLES]

1. Let X be a topological space and for every open subset $U \subset X$, let $\mathcal{F}(U)$ be the continuous functions from $U \to \mathbb{R}$ and let ρ be the restriction maps.
2. Let X be a smooth manifold and $U \subset X$ be open. Then, if we consider $\Omega^k(U)$, the smooth-k forms on U with ρ being the restriction, these form a sheaf.
3. Let X be a variety over a field k equipped with the Zariski Topology. For each $U \subseteq X$ open, let $\mathcal{C}(U)$ be the ring of regular functions from $U \to k$ and $\rho_{UV} : \mathcal{C}(U) \to \mathcal{C}(V)$ is the standard restriction map. To check the Sheaf conditions, we note that the a regular function which is locally 0 is just 0 (regular maps are polynomial maps) and a function which is locally regular is regular. We call this the sheaf of regular functions on X.
4. (Constant Sheaf) X a topological space and A an abelian group. Give A the discrete topology and for any open set $U \subset X$ let $\mathcal{A}(U)$ be the group of all continuous maps from $U \to A$. Then under the standard restriction map, this is a sheaf called "The Constant Sheaf" (This is apparently because for every open connected set U, $\mathcal{A}(U) \cong \mathcal{A}$.

However, note that not every Presheaf is a sheaf. We look at

EXAMPLE **0.7.** Let $X = \mathbb{R}$ with the standard Euclidean topology and for an open set $U \subset X$ define $\mathcal{F}(U) = \{f : U \to \mathbb{R} : f(x) = c \text{ for some } c \in \mathbb{R}\}$ and where the restriction maps are jus the restrictions of the functions. We leave it to the reader to check that this is indeed a Presheaf. However it is not a Sheaf because it fails the Gluing condition. Let $U = (0,1) \cup (2,3)$. Define $s_1 \in \mathcal{F}((0,1))$ and $s_2 \in \mathcal{F}((2,3))$ by putting $s_1 = 1$ and $s_2 = 2$. Then, we cannot find $s \in \mathcal{F}(U)$ such that the restriction to $(0,1)$ is 1 and the restriction to $(2,3)$ is 2. The main reason why this fails is because Sheaves are supposed to capture local properties but being constant globally is not a local property. If instead we changed our sections so that they consisted of functions which were locally constant, then we could glue them together and it would be a Sheaf.

DEFINITION **0.8.** If $\mathcal{F}$,$\mathcal{G}$ are presheaves on X, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves then we use the same definition for a morphism of Sheaves. An isomorphism is a morphism which has a two sided inverse. What the above diagram is really saying is that a morphism of Presheaves preservers the local structure, i.e. taking image under the morphism and then restricting is the same as restricting and then taking the image under φ. Moreover, the isomorphism property basically amounts to saying that $\varphi(U)$ is an isomorphism for every open subset U of X.

REMARK **0.9.** The morphisms from a Sheaf $\mathcal{F}$ to $\mathcal{G}$ are precisely the morphisms from $\mathcal{F}$ to $\mathcal{G}$ as presheaves. That is, the category of Sheaves is a full subcategory of the category of presheaves on X. If you interpret a presheaf as a contravariant functor from the category of open sets, a morphism of presheaves is a natural transformation of functors.

EXAMPLE **0.10.** Let $\mathcal{F}$ be the sheaf of differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ and let $\mathcal{G}$ be the sheaf of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Since any differentiable function on an open set U can be viewed as a continuous function on U, one may check that we obtain a morphism of sheaves.

The next thing we are going to be talking about are Stalks of Presheafs at a point p. These will be covered in greater detail in the following lecture. There are 2 equivalent ways to define a stalk, the more abstract one using the language of category theory will be discussed next time. The other method is more concrete and I present here the notion of a Stalk to emphasize the point that the structures we are developing right now really want to capture local structure.

DEFINITION **0.11.** [GERM OF A DIFFERENTIABLE FUNCTION] Equip the set $\{(f, U) : U \text{ open }, p \in U, f \in \mathcal{F}(U)\}$ with the equivalence relation given by $(f, U) \sim (g, V)$ iff there exists $W \subset U \cap V$

with $p \in W$ such that $f|_W = g|_W$. That is, locally at p, the two functions behave the same. An equivalence class is called a germ at p. We call the set of germs at p, the stalk at p.

Definition **0.12.** [Stalk] We define the Stalk of a Presheaf $\mathcal{F}$ at the point p to be the set $\{(f, U) : f \in \mathcal{F}(U)\}$ modulo the equivalence relation defined by $(f, U) \sim (g, V)$ iff there is some open set $W \subset U, V$ with $p \in W$ such that $f|_W = g|_W$.

remark **0.13.** Note that stalk at p actually forms a ring. One can check that given two germs, f, g they can be added and this addition is well defined, etc. In coming lectures, the stalk will be defined more abstractly using the language of category theory, but going forward this might be a useful equivalent definition to keep in mind. Also observe that Germs are fundamentally capturing the local property of our differentiable function from the motivating example.