AGS - LECTURE 2

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Definition 0.1. A map $f : V \subset \mathbb{A}^n \to W \subset \mathbb{A}^m$, where V and W are algebraic sets is a *polynomial* map if there exist $p_1, \ldots, p_m \in A$ such that $f(a) = (p_1(a), \ldots, p_m(a))$

We observe that given a polynomial map $\varphi : V \to W$, we obtain a natural map $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$ given by $\varphi(p) = p \circ \varphi$. This is in fact a k-algebra homomorphism. Moreover, all k-algebra homomorphisms arise in this manner, and so we obtain a bijective correspondence between k-algebra homomorphisms $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$ and polynomial maps $\varphi : V \to W$.

1. The Local Ring

Definition 1.1. Let D be an integral domain. We define its *field of fractions*, $K(D)$, to be the set $D \times D$ modulo the following equivalence relation:
$(a, b) \cong (c, d)$ if and only if $ad = bc$

Given a variety V and a point $p \in V$, we define the *local ring* at p, $\mathcal{O}_p(V)$ to be the set of $f \in K(\Gamma(V))$ such that $f = \frac{a}{b}$ for some $a, b \in \Gamma(V)$ with $b(p) \neq 0$. When such a representative exists, we say that f is *defined* at p.

We may also define $\mathcal{O}(V) = \cap_p \mathcal{O}_p(V)$, the set of all elements of the co-ordinate ring defined on V. It turns out that $\mathcal{O}(V) = \Gamma(V)$.

Definition 1.2. Let R be an integral domain. Suppose $S \subset R$ is a set closed under multiplication. Then the *localization* of R at S, denoted $S^{-1}R$ is the set of $f \in K(R)$ such that $f = \frac{a}{b}$ for some $a \in R$, $b \in S$. It is clear that for $S^{-1}R$ to be a subring of $K(R)$, it is sufficient that S be multiplicatively closed. If $\mathfrak{p}$ is a prime ideal, then by the *localization at* $\mathfrak{p}$, we mean $(\mathfrak{p}^c)^{-1}R$.

It is clear that when taking the localization at a prime ideal, we obtain a ring with a unique maximal ideal: the set of non-units. The local ring $\mathcal{O}_p(V)$ is the localization of $\Gamma(V)$ at the set of co-ordinate polynomials that vanish at p.

2. Dimension

Definition 2.1. Let R be a ring. We define the *Krull Dimension* to be the longest natural number n such that there exists a chain of prime ideals

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n = R$$

Note that we can define a corresponding value for prime ideals, which we refer to as the height of the prime ideal.

We define the Krull dimension of a variety to be the longest chain of proper subvarieties.

$$V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$$

Proposition 2.2. *Let V be an affine variety. The following are equal:*

(1) *The Krull dimension of V*
(2) *The Krull dimension of $\Gamma(V)$.*
(3) *The transcendence degree of $K(\Gamma(V))$ over k.*

3. Projective Varieties

Definition 3.1. We define *Projective n-Space* to be the set of 1-dimensional subspaces of $\mathbb{A}^{n+1}$.

That is, we take $\mathbb{P}^n = \mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\} / =$

Where $(a_0, \ldots, a_n) = (b_0, \ldots, b_n)$ if for some fixed $\lambda \in k$ we have $a_i = \lambda b_i$ for all i.

Remark 3.2. Projective space is a kind of compactification of $\mathbb{A}^{n+1}$

To attempt to recreate algebraic geometry in Projective space, we will need to consider special ideals.

Definition 3.3. A polynomial is *homogeneous* if every term has the same degree.

An ideal is *homogeneous* if it is generated by a collection of homogeneous polynomials.

Theorem 3.4 (Projective Nullstellensatz)**.** *There is a 1-1, inclusion reversing correspondence between projective algebraic sets in $\mathbb{P}$ and homogeneous radical ideals of $k[x_0, \ldots, x_n]$ not equal to I_+, the ideal generated by homogeneous polynomials of degree greater than 0.*

Remark 3.5. In a similar vein to the affine case, projective varieties correspond to prime homogeneous ideals, and we define the co-ordinate ring of V to be $k[x_0, \ldots, x_n]/V(X)$