The affine ideal-variety correspondence

Michael L. Baker

University of Waterloo

mlbaker.org

May 14, 2013

Overview

1. (Affine) algebraic sets
2. Ideals and coordinate rings

Introduction

Let k be a field (often assumed to be algebraically closed). We want to talk about subsets of space that are somehow "carved out" by polynomials, that is, *solution sets* of polynomial systems.

In general, we will restrict ourselves to studying polynomial (and rational) phenomena. This will dictate many of the definitions, such as which objects and morphisms we care about. As a result, the nature of algebraic geometry has a certain "finiteness" to it.

Affine n-space and the polynomial ring

Let k be a field (often assumed to be algebraically closed).

Definition

Affine n**-space** over k is defined by

$$\mathbf{A}_k^n = \mathbf{A}^n = \{\text{points } P = (a_1, \ldots, a_n) : \text{each } a_i \in k\} = k^n.$$

Let $A = k[x_1, \ldots, x_n]$ be the **polynomial ring** in n variables over k.

Remark

Elements of A are interpreted as functions from $\mathbf{A}^n \to k$: we can *evaluate* a polynomial $f \in A$ at any point $P \in \mathbf{A}^n$.

Viewing a commutative ring as "functions on some space" can be very insightful!

Zero sets of polynomials

Definition

We can talk about the set of common zeros (that is, the **zero set**) of any family $S \subseteq A$ of polynomials:

$$Z(S) = \{P \in \mathbf{A}^n : f(P) = 0 \text{ for all } f \in S\} = \bigcap_{f \in S} Z(f)$$

(when S is finite, e.g. a single polynomial, we omit the $\{\ \}$'s). Zero sets of single non-constant polynomials are called **hypersurfaces**.

Remark

It's easy to prove that if $I = (S)$, that is I is the smallest ideal containing S, then $Z(I) = Z(S)$. Moreover, if $S \subseteq T$ then $Z(S) \supseteq Z(T)$ (inclusions are reversed). Finally, the **Hilbert Basis Theorem** says A is Noetherian (= every ideal is finitely generated), so in fact *we can always arrange for S to be finite*!

Affine algebraic sets

Definition

A subset $X \subseteq \mathbf{A}^n$ is called an **(affine) algebraic set** if $X = Z(S)$ for some $S \subseteq A$.

Not all subsets of $\mathbf{A}^n$ are algebraic (e.g. sine curve).

Proposition

- If $S, T \subseteq A$, then $Z(S) \cup Z(T) = Z(ST)$, where
$$ST = \{fg : f \in S, g \in T\}.$$
- If $\{S_\alpha\} \subseteq A$, then
$$\bigcap_\alpha Z(S_\alpha) = Z(\bigcup_\alpha S_\alpha).$$
- $\varnothing = Z(1)$ and $\mathbf{A}^n = Z(0)$.

Examples of algebraic sets

Point

If $P = (a_1, \ldots, a_n) \in \mathbf{A}^n$, then

$$\{P\} = Z(x_1 - a_1, \ldots, x_n - a_n).$$

Circle

In $\mathbf{A}^2_{\mathbb{R}}$, the unit circle S^1 is given by

$$S^1 = Z(x^2 + y^2 - 1).$$

"Twisted Cubic"

In $\mathbf{A}^3_{\mathbb{R}}$, the **twisted cubic** $\mathcal{T} = \{(t, t^2, t^3) : t \in \mathbb{R}\}$ is given by

$$\mathcal{T} = Z(y - x^2, z - x^3).$$

Zariski topology

Definition

We saw finite unions and arbitrary intersections of algebraic sets are algebraic, as are $\varnothing$ and $\mathbf{A}^n$. This means they form the *closed sets of a topology* on $\mathbf{A}^n$. It is called the **Zariski topology**.

Remarks

The Zariski topology is almost never Hausdorff (this happens only when k is finite, but recall finite fields are never alg. closed). In particular it is not metric. Be very wary of it: things like connectedness in this topology will not match your Euclidean intuition!

Exercise

What is the Zariski topology on $\mathbf{A}^1$? Remember, $A = k[x]$ is a *principal ideal domain* and polynomials in one variable have finitely many roots.

Irreducibility

Definition

A nonempty subset Y of a topological space X is **irreducible** it cannot be expressed as $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. $\varnothing$ is not considered irreducible.

An **affine algebraic variety** (or just **affine variety**) is an irreducible closed subset of the topological space $(\mathbf{A}^n, \text{Zariski})$. That is, it's an irreducible algebraic set. An open subset of an affine variety is called a **quasi-affine variety**.

Remark

Every algebraic set can be decomposed into a union of affine varieties ("irreducible components"), no one containing another, unique up to re-ordering.

Ideal of a subset

From algebra (ideals of A) we obtained geometry (subsets of $\mathbf{A}^n$). Now, can we go the other way? Given an algebraic set $X \subseteq \mathbf{A}^n$, how do we define the "ring of polynomials on X"?

Definition

Let $X \subseteq \mathbf{A}^n$. The **ideal** of X is the set of all polynomials vanishing on all of X:

$$I(X) = \{f \in A : f(P) = 0 \text{ for all } P \in X\} \subseteq A.$$

Convince yourself that it's indeed an ideal of the polynomial ring A.

Curious property

If $f \in A$ is such that f^n vanishes on X, then so too must f (why?). This reads "$f^n \in I(X)$ implies $f \in I(X)$".

Radical ideals

Motivated by the "curious property" we remarked about ideals of the form $I(X)$, we make the following definition.

Definition

If I is an ideal of a ring R, the **radical** of I is given by

$$\sqrt{I} = \mathrm{Rad}(I) := \{r \in R : r^n \in I \text{ for some } n > 0\}.$$

An ideal such that $I = \sqrt{I}$ is called a **radical ideal**.

Remark

Hence, the ideal $I(X)$ of any $X \subseteq \mathbf{A}^n$ is a radical ideal. Also, an algebraic set $X \subseteq \mathbf{A}^n$ is irreducible if and only if $I(X)$ is *prime*. All prime ideals are radical, since any integral domain certainly admits no nilpotent elements!

Hilbert's Nullstellensatz

We observed that the ideal of a set of points is always radical. It would be great if every radical ideal was the ideal of *some set of points*. Hilbert showed that if k is alg. closed, this is indeed true!

Theorem (Hilbert's Nullstellensatz)

Let k be alg. closed, and $I \subseteq A$ be an ideal. Then $I(Z(I)) = \sqrt{I}$.

Remark

As desired, if I is already radical, the above assures us that I is indeed the ideal of $Z(I)$.

Example

It is crucial that k be algebraically closed; consider $k = \mathbb{R}$ and $I = (x^2 + 1)$. $x^2 + 1$ has *no roots*, so the Nullstellensatz yields $A \supsetneq (x^2 + 1) = I(\varnothing) = A$ which is nonsense: indeed $x \notin (x^2 + 1)$.

Ideal-variety correspondence*

$I \mapsto Z(I)$ sends ideals of A (algebra) to algebraic subsets of $\mathbf{A}^n$ (geometry). Conversely, $X \mapsto I(X)$ sends subsets of $\mathbf{A}^n$ to radical ideals of A.

Properties

- If $S_1 \subseteq S_2 \subseteq A$ then $Z(S_1) \supseteq Z(S_2)$.
- If $X_1 \subseteq X_2 \subseteq \mathbf{A}^n$ then $I(X_1) \supseteq I(X_2)$.
- If $X_1, X_2 \subseteq \mathbf{A}^n$ then $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
- If $I \subseteq A$ is an ideal, then $I(Z(I)) = \sqrt{I}$ (this is the Nullstellensatz).
- If $X \subseteq \mathbf{A}^n$, $Z(I(X)) = \overline{X}$, the Zariski closure of X, that is, the smallest algebraic set containing X.

Thus this is a *one-to-one, inclusion-reversing correspondence* between algebraic sets in $\mathbf{A}^n$ and radical ideals in A.

(*) This is an unfortunate name, but it sounds better than "ideal-algebraic set correspondence".

Some examples

From Hartshorne (note carefully that he always assumes k is algebraically closed):

- $\mathbf{A}^n$ is irreducible since it corresponds to $(0) \subseteq A$, a prime ideal.
- An irreducible polynomial $f \in k[x, y]$ generates a prime ideal in A since A is a *unique factorisation domain*, so $X = Z(f)$ is irreducible. X is called the **affine curve** defined by f; we say $\deg f$ is its **degree**.
- Maximal ideals of A correspond to minimal irreducible algebraic sets ($=$ points, by Nullstellensatz).

Coordinate rings

We now define the "ring of polynomials on an algebraic set", answering an earlier question.

Idea

If $X \subseteq \mathbf{A}^n$ is algebraic, the polynomials on X should just be the polynomials on $\mathbf{A}^n$, with the exception that two polynomials are considered the same if they *agree* on X. Now note f and g agree on X if and only if their difference $f - g$ *vanishes* on X!

Definition

If $X \subseteq \mathbf{A}^n$ is algebraic, we define the **coordinate ring** of X to be the quotient ring

$$A(X) = \Gamma(X) := \frac{A}{I(X)}.$$

We'll see that "nice" maps between varieties induce maps between their coordinate rings.

Coordinate rings are k-algebras

A non-zero constant polynomial will not vanish on X, so the natural quotient map $\pi : A \twoheadrightarrow A(X)$ restricts to an embedding $k \hookrightarrow A(X)$.

If X is a variety then $I(X)$ is prime, so $A(X)$ is an integral domain with a copy of k sitting inside it. These are called k-**algebras**. To be extremely specific, $A(X)$ is a *finitely generated k-algebra* which is also an integral domain.

Punch line: This specific category of k-algebras is somehow “roughly the same thing” as the category of affine varieties!