Bundles, spheres, and Clifford algebras

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Abstract

After introducing the concept of a vector bundle, we will examine the problem of how many linearly independent vector fields can be found on the n-sphere S^n, and discuss how representations of Clifford algebras can be used to give a construction. Frank Adams later proved using deep methods of K-theory that this construction in fact gives the largest possible number of such vector fields.

Bundles

Given a (smooth) manifold M, we can make sense of what it means for a function $f : M \to \mathbf{R}$ to be smooth. So we obtain the $\mathbf{R}$-algebra of all such smooth functions, $C^\infty(M)$. However, we may also want to consider other kinds of (say, vector-valued) functions on M. Often it is the case that, to each point of M there is attached some fixed "model space" F, and the totality of these spaces itself, which we denote E, carries a natural manifold structure. This is roughly the concept of a fibre bundle. The trivial example is that F is some manifold, and we consider the product manifold, $E := M \times F$. For example, if $F = M = S^1$ then we obtain $S^1 \times S^1$, the standard torus. We have taken a circle and placed another circle above all of its points, in the most straightforward way possible. However, we could also have inserted a "twist" and obtained a more interesting fibre bundle, the *Klein bottle*. ...

Review

Recall the following notation from last time. Let V be an n-dimensional vector space over $\mathbf{R}$, equipped with a non-degenerate quadratic form q. We can choose a basis for $V \cong \mathbf{R}^n$ such that

$$q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2$$

where $r + s = n$ and $0 \leq r \leq n$. We now introduce a bunch of simplified notation: write $q_{r,s} = q$, $\mathrm{O}_{r,s} = \mathrm{O}(V, q)$, $\mathrm{SO}_{r,s} = \mathrm{SO}(V, q)$, and

$$\mathrm{Pin}_{r,s} = \mathrm{Pin}(V, q), \qquad \mathrm{Spin}_{r,s} = \mathrm{Spin}(V, q).$$

Similarly we write

$$\mathrm{O}_n = \mathrm{O}_{n,0} \cong \mathrm{O}_{0,n}, \qquad \mathrm{SO}_n = \mathrm{SO}_{n,0} \cong \mathrm{SO}_{0,n},$$
$$\mathrm{Pin}_n = \mathrm{Pin}_{n,0}, \qquad \mathrm{Spin}_n = \mathrm{Spin}_{n,0},$$
$$\mathrm{P}_{r,s} = \mathrm{P}(V, q), \qquad \widetilde{\mathrm{P}}_{r,s} = \widetilde{\mathrm{P}}(V, q),$$

and note from the paragraph above that

$$\mathrm{P}_{r,s} = \widetilde{\mathrm{P}}_{r,s}.$$

We now study the Clifford algebras $\mathcal{C}l_{r,s} = \mathcal{C}l(V, q)$ where $V = \mathbf{R}^{r+s}$ and

$$q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2.$$

Of course we are particularly interested in the cases $\mathcal{C}l_n = \mathcal{C}l_{n,0}$ and $\mathcal{C}l_n^* = \mathcal{C}l_{0,n}$.

Remark. The algebra $\mathcal{C}l_{r,s}$ contains the groups $\mathrm{Spin}_{r,s}$ and $\mathrm{Pin}_{r,s}$, and so any representation of the algebra $\mathcal{C}l_{r,s}$ restricts to a representation of these groups which is non-trivial on the element -1; such representations are therefore *not* pullbacks of representations of $\mathrm{O}_{r,s}$ or $\mathrm{SO}_{r,s}$.

Proposition ($\star$). Let $e_1, \ldots, e_{r+s}$ be any q-orthonormal basis of $\mathbf{R}^{r+s} \subset \mathcal{C}l_{r,s}$. Then $\mathcal{C}l_{r,s}$ is generated (as an algebra) by $e_1, \ldots, e_{r+s}$ subject to the relations

$$e_i e_j + e_j e_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \\ +2\delta_{ij} & \text{if } i > r. \end{cases}$$

Proof. This is pretty clear. $\square$

Under the canonical isomorphism $\mathcal{C}l_n \cong \Lambda^* \mathbf{R}^n$, Clifford multiplication has a nice interpretation. Using the inner product on $\mathbf{R}^n$ we can identify $\mathbf{R}^n$ with its dual. We can thereby talk about the **interior product** or **contraction** in $\Lambda^* \mathbf{R}^n$. For $v \in \mathbf{R}^n$, this is a linear map $(v \llcorner) : \Lambda^p \mathbf{R}^n \to \Lambda^{p-1} \mathbf{R}^n$ given on simple vectors by

$$v \llcorner (v_1 \wedge \ldots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} \langle v_i, v \rangle v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_p$$

where $\hat{}$ indicates deletion. This gives a skew-derivation of the algebra, that is,

$$v \llcorner (\varphi \wedge \psi) = (v \llcorner \varphi) \wedge \psi + (-1)^p \varphi \wedge (v \llcorner \psi)$$

for any $\varphi \in \Lambda^p \mathbf{R}^n$. It is not difficult to see that $v \llcorner (v \llcorner) = 0$ for any $v \in \mathbf{R}^n$. Hence, by universality the interior product extends to all elements of $\Lambda^* \mathbf{R}^n$, i.e. to a bilinear map $\Lambda^* \mathbf{R}^n \times \Lambda^* \mathbf{R}^n \to \Lambda^* \mathbf{R}^n$.

Proposition. With respect to the canonical isomorphism $\mathcal{C}l_n \cong \Lambda^* \mathbf{R}^n$, Clifford multiplication between $v \in \mathbf{R}^n$ and any $\varphi \in \mathcal{C}l_n$ can be written as

$$v \cdot \varphi \cong v \wedge \varphi - v \llcorner \varphi$$

Classification of Clifford algebras

Before we proceed, we will need some results from the classification of Clifford algebras. In particular:

Theorem. For all $k, \ell \geq 0$, we have isomorphisms

$$\begin{aligned} \mathcal{C}l_{0,k+2} &\cong \mathcal{C}l_{k,0} \otimes \mathcal{C}l_{0,2} \\ \mathcal{C}l_{k+2,0} &\cong \mathcal{C}l_{0,k} \otimes \mathcal{C}l_{2,0} \\ \mathcal{C}l_{k+1,\ell+1} &\cong \mathcal{C}l_{k,\ell} \otimes \mathcal{C}l_{1,1}. \end{aligned}$$

Proof. Let $e_1, \ldots, e_{k+2}$ be an orthonormal basis for $\mathbf{R}^{k+2}$ in the standard inner product, and let $q(x) = -\|x\|^2$. Let $e_1', \ldots, e_k'$ denote standard generators for $\mathcal{C}l_{k,0}$ and let e_1'', e_2'' denote standard generators for $\mathcal{C}l_{0,2}$ (in the sense of $\star$). Define a map $f : \mathbf{R}^{k+2} \to \mathcal{C}l_{k,0} \otimes \mathcal{C}l_{0,2}$ by setting

$$f(e_i) = \begin{cases} e_i' \otimes e_1'' e_2'' & \text{for } 1 \leq i \leq k \\ 1 \otimes e_{i-k}'' & \text{for } i = k+1,\ k+2 \end{cases}$$

and extending linearly. Note that for $1 \leq i, j \leq k$, we have

$$f(e_i) f(e_j) + f(e_j) f(e_i) = (e_i' e_j' + e_j' e_i') \otimes (-1) = 2\delta_{ij} 1 \otimes 1;$$

and for $k+1 \leq \alpha, \beta \leq k+2$ we have

$$f(e_\alpha)f(e_\beta) + f(e_\beta)f(e_\alpha) = 1 \otimes (e''_{\alpha-k}e''_{\beta-k} + e''_{\beta-k}e''_{\alpha-k}) = 2\delta_{\alpha\beta} 1 \otimes 1.$$

Also we see that

$$f(e_i)f(e_\alpha) + f(e_\alpha)f(e_i) = 0.$$

It follows that $f(x)f(x) = \|x\|^2 1 \otimes 1$ for all $x \in \mathbf{R}^{k+2}$. Hence, by the universal property, f extends to an algebra morphism $\widetilde{f} : \mathcal{C}l_{0,k+2} \to \mathcal{C}l_{k,0} \otimes \mathcal{C}l_{0,2}$. Since $\widetilde{f}$ maps onto a set of generators for $\mathcal{C}l_{k,0} \otimes \mathcal{C}l_{0,2}$, it must be surjective. Then, since $\dim \mathcal{C}l_{0,k+2} = \dim \mathcal{C}l_{k,0} \otimes \mathcal{C}l_{0,2}$, we conclude that $\widetilde{f}$ must be an isomorphism. This proves the first isomorphism. The second is entirely analogous; the third is not really needed for us and is left as an exercise. □

We need the following elementary facts concerning the tensor products of algebras over $\mathbf{R}$.

Proposition (tensor products of algebras).

$$\begin{aligned}
(\mathbf{R} \times \mathbf{R}) \otimes A &\cong A \times A \\
\mathbf{R}(n) \otimes A &\cong A(n) \\
\mathbf{H} \otimes \mathbf{C} &\cong \mathbf{C}(2) \\
\mathbf{R}(n) \otimes \mathbf{R}(m) &\cong \mathbf{R}(nm) \\
\mathbf{H} \otimes \mathbf{H} &\cong \mathbf{R}(4).
\end{aligned}$$

Using these, we can build the table:

Constructing the vector fields

It turns out that by understanding the representations of the algebras $\mathcal{C}l_k$, we can construct linearly independent vector fields on spheres.

Definition (representation/module for $\mathcal{C}l_k$). A **representation** of $\mathcal{C}l_k$ (or a $\mathcal{C}l_k$**-module**) V is just a real vector space with an $\mathbf{R}$-algebra morphism $\mathcal{C}l_k \to \mathrm{End}_{\mathbf{R}} V$. That is, there is a multiplication $(\varphi, v) \mapsto \varphi \cdot v$ for $\varphi \in \mathcal{C}l_k$ and $v \in V$, which we call **Clifford multiplication**.

Definition. Define $F_k \subset \mathcal{C}l_k^\times$ to be the finite group generated by an orthonormal basis $e_1, \dots, e_k$ of $\mathbf{R}^k$.

The group F_k can be presented by the abstract elements $e_1, \dots, e_k, -1$ subject to the relation that -1 is central and $(-1)^2 = 1$, $e_i^2 = -1$ and $e_i e_j = (-1) e_j e_i$ for all $i \neq j$. The Clifford algebra is nearly the group algebra of F_k, in the sense that

$$\mathcal{C}l_k \cong \mathbf{R}F_k / \mathbf{R} \cdot \{(-1) + 1\}.$$

It is clear that representations of $\mathcal{C}l_k$ correspond exactly to linear representations of F_k such that -1 acts by $-\mathrm{id}$. This group yields the following important conclusion.

Proposition (unitarizability). Let $\mathcal{C}l_k \to \mathrm{Hom}_{\mathbf{R}}(W, W)$ be a real representation of $\mathcal{C}l_k$. Then there exists an inner product $\langle \bullet, \bullet \rangle$ on W such that Clifford multiplication by unit vectors $e \in \mathbf{R}^k$ is orthogonal, that is,

$$\langle e \cdot w, e \cdot w' \rangle = \langle w, w' \rangle$$

for all $w, w' \in W$ and for all $e \in \mathbf{R}^k$ with $\|e\| = 1$.

Proof. Choose an arbitrary inner product and average it over the finite group F_k. Note that if $e = \sum a_j e_j$ where $\sum a_j^2 = 1$, then

$$\langle ew, ew \rangle = \sum a_j^2 \langle e_j w, e_j w \rangle + \sum_{i \neq j} a_i a_j \langle e_i w, e_j w \rangle = \langle w, w \rangle$$

since $\langle e_i w, e_i w \rangle = \langle w, w \rangle$ and for $i \neq j$, $\langle e_i w, e_j w \rangle = \langle e_j e_i w, -w \rangle = \langle e_i e_j w, w \rangle = -\langle e_j w, e_i w \rangle = 0$. □

Corollary (skew-symmetry of Clifford multiplication). In this inner product $\langle \bullet, \bullet \rangle$, Clifford multiplication by any vector $v \in \mathbf{R}^k$ is a skew-symmetric transformation of W. That is,

$$\langle v \cdot w, w' \rangle = -\langle w, v \cdot w' \rangle$$

for any $w, w' \in W$.

Proof. Assume $v \neq 0$. Then

$$\langle v \cdot w, w' \rangle = \langle (v/\|v\|) \cdot v \cdot w, (v/\|v\|) \cdot w' \rangle = (1/\|v\|^2) \langle v^2 \cdot w, v \cdot w' \rangle = -\langle w, v \cdot w' \rangle. \qquad \square$$

Proposition (construction of vector fields). Suppose $\mathbf{R}^n$ is a module for the Clifford algebra $\mathcal{C}l_k$. Then there exist k pointwise linearly independent tangent vector fields on the sphere S^{n-1} and also on the projective space $\mathbf{P}^{n-1}(\mathbf{R}) = S^{n-1}/\mathbf{Z}_2$.

Proof. Choose an inner product in $\mathbf{R}^n$ so that Clifford multiplication by unit vectors in $\mathbf{R}^k$ is orthogonal (see above proposition). Let

$$S^{n-1} = \{x \in \mathbf{R}^n : \|x\|^2 = 1\}.$$

Choose a basis $v_1, \ldots, v_k$ for $\mathbf{R}^k$, and to each v_j associate the vector field V_j on $\mathbf{R}^n$ defined by

$$V_j(x) = v_j \cdot x \qquad j = 1, \ldots, k$$

where the dot denotes Clifford multiplication. Since the linear transformation $x \mapsto v \cdot x$ is skew-symmetric (see above), we have that $\langle V_j(x), x \rangle = \langle v_j x, x \rangle = 0$. Hence, the vector fields V_j are tangent to S^{n-1}. It remains to show that $V_1, \ldots, V_k$ are pointwise linearly independent. Fix $x \in S^{n-1}$ and consider the linear map $i_x : \mathbf{R}^k \to T_x S^{n-1} \subset \mathbf{R}^n$ given by

$$i_x(v) = v \cdot x$$

The image of i_x is the linear span of $V_1(x), \ldots, V_k(x)$, so it suffices to prove that i_x is injective. However if $i_x v = v \cdot x = 0$, then $v \cdot v \cdot x = -\|v\|^2 x = 0$ and so $v = 0$. Since $V_j(-x) = -V_j(x)$, these vector fields descend to (pointwise linearly independent) vector fields on $\mathbf{P}^{n-1}(\mathbf{R})$. $\square$

The question therefore becomes: given an integer n, what is the largest number of independent vector fields on S^{n-1} that can be constructed in this manner? That is, what is the largest integer k such that $\mathbf{R}^n$ is a $\mathcal{C}l_k$-module? We recall that the dimension of an irreducible $\mathcal{C}l_k$-module is always a power of 2. Hence we want to find the largest power of 2 which divides n. That is, we write $n = p2^m$ where p is odd, then we consult the table to find the largest k such that $d_k = 2^m$. The result is the following.

Theorem (Radon-Hurwitz-Eckmann). On the sphere S^{n-1} (and on the projective space $\mathbf{P}^{n-1}(\mathbf{R})$) there exist k pointwise linearly independent vector fields where k is calculated as follows. Write $n = 2^{4a+b}m$, $0 \leq b \leq 3$, and m odd. Then

$$k = 8a + 2^b - 1.$$

Remark. The number $8a + 2^b$ is usually written $\rho(n)$. These numbers $\rho(n)$ are called the **Radon-Hurwitz numbers**.

Proof. One need only check this when $a = 0$, and then note that for each increase of k by 8 the dimension of the vector space for an irreducible representation of $\mathcal{C}l_k$ increases by 2^4. Note that when n is odd (i.e. we are on an even sphere) the number of such vector fields is zero as it must be, since the Euler characteristic is nonzero in this case. Note also that this construction gives three vector fields on S^3, seven on S^7 and eight on S^{15}. $\square$

Here is a striking result of algebraic topology, whose proof relies on deep methods from topological K-theory.

Theorem (J.F. Adams). The number of vector fields constructed on S^{n-1} above is the largest possible number of linearly independent vector fields that can exist on S^{n-1}!

The following corollary was actually known before the above theorem was proved.

Corollary (Kervaire, c. 1956). A sphere is said to be **parallelizable** when its tangent bundle is trivial, i.e. $\rho(n) = n$. This occurs exactly when $n \in \{1, 2, 4, 8\}$, so the only spheres which have trivial tangent bundles are S^0, S^1, S^3, and S^7.