Clifford algebras and spin representations I

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Abstract

The Clifford algebra $\mathcal{C}l(V, q)$ associated to a quadratic vector space (V, q) is a quantization of its exterior algebra $\Lambda(V)$ whose multiplication carries very rich geometric information. Now an indispensable tool in geometry and mathematical physics, these algebras are also intimately related to projective representations of the orthogonal Lie groups. Continuing from last time, I will develop their basic theory: the universal property, the natural filtration, the groups Pin and Spin, the twisted adjoint representation, and so on. We will see a few examples in simple cases.

I mostly follow (rather closely) the first chapter of Lawson and Michelsohn's excellent book, *Spin Geometry*. I apologize in advance for any excessive similarities; I often found it remarkably difficult to improve upon such masterful exposition.

Motivation

Given a (connected) Lie group G, its Lie algebra $\mathfrak{g}$ is a crucial tool for studying its structure. However, one should remember that the representation theory of $\mathfrak{g}$ generally does not correspond to that of G, but rather to that of its universal covering group $\widetilde{G}$. Of course if G is simply connected, then $G = \widetilde{G}$, so life is good. This is why all the information we obtained about the representations of $\mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{sl}(3, \mathbf{C})$ can be translated easily to information about the representations of $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$: $\mathfrak{sl}(n, \mathbf{C})$ is the complexification of the *real* Lie algebra $\mathfrak{su}(n)$, and we have a correspondence theorem there.

Hence, when considering the Lie algebras $\mathfrak{so}(n, \mathbf{C})$, we are not really picking up the representation theory of $\mathrm{SO}(n)$, but rather of the universal covering group $\mathrm{Spin}(n)$, which double covers $\mathrm{SO}(n)$. One of our first tasks will be to show that we have the exact sequence:

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{Spin}(n) \xrightarrow{\pi} \mathrm{SO}(n) \to 1.$$

Of course, many of the representations of $\mathrm{Spin}(n)$ arise simply from pulling back representations of $\mathrm{SO}(n)$ along the covering map π. Clearly the negative identity element of $\mathrm{Spin}(n)$ will act trivially in any such representation, so perhaps unsurprisingly, it turns out that *some* irreps of $\mathrm{Spin}(n)$, called the *spin representations*, cannot be obtained this way.

A "spinor" is (very roughly) an element of one of these "spin" representation spaces; one of my goals for this last month of the seminar is to understand *what*, exactly, spinors are.

> "No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as the understanding of the square root of -1 took centuries, the same might be true of spinors." – Sir Michael Atiyah.

There are some low-dimensional isomorphisms, for example $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong S^3$, the unit sphere of the quaternions (as is probably well-known to most of you), and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Clifford algebras (in particular, their classification) are the main tool that will help us construct and understand these spin groups and their spin representations. If we relax the concept of a representation (morphism $G \to \mathrm{GL}(V)$) to that of a so-called "projective representation", i.e. morphism $G \to \mathrm{PGL}(V)$, we can still "see" these spin representations at the level of $\mathrm{SO}(n)$. In a projective representation we need not have $\rho(gh) = \rho(g)\rho(h)$, but rather only that this equation hold true up to multiplication by a scalar. Projective representations, in general, are closely related to linear ("true") representations of covering groups, or more generally, central extensions.

Keeping in mind that orthogonal groups come from quadratic forms on V (which, in characteristic 0, are equivalent to symmetric bilinear forms), it seems like we should attempt to better understand quadratic forms. The exterior algebra of V

$$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k V$$

clearly keeps track of no "geometry"; it is constructed independently of any quadratic form on V. The idea of Clifford algebras is to "deform", or "quantize" the multiplication of $\Lambda(V)$, using a quadratic form on V. In fact, we will see that $\Lambda(V)$ is the Clifford algebra associated to the "totally isotropic" (that is, identically zero) quadratic form on V.

Every Clifford algebra on V is naturally isomorphic *as a vector space* (not as an algebra) to $\Lambda(V)$, but the multiplication is strictly richer!

Clifford algebras

Let V be a vector space over $K = \mathbf{R}$ or $\mathbf{C}$, always assumed finite-dimensional (the theory of Clifford algebras *can* be carried out for modules over more general rings, however). By **algebra**, we mean an associative K-algebra with 1. We first recall the definition of a quadratic form.

Definition. A **quadratic form** on V is a map $q : V \to K$ such that:

- for all $c \in K$ and $v \in V$, we have $q(cv) = c^2 v$.
- $2q(v, w) := q(v + w) - q(v) - q(w)$ is a symmetric bilinear form.

We say the pair (V, q) is a **quadratic space**.

Which morphisms do we care about between quadratic spaces? If (V, q) and (V', q') are quadratic spaces, then naturally we want to consider linear maps $f : V \to V'$ such that $f^*q' = q$, where f^*q' is the pullback of the quadratic form q' along f. That is to say $q'(fv) = q(v)$ for all $v \in V$. In this way, we obtain a category; in fact, **orthogonal groups** arise precisely as the automorphism groups of objects in this category, that is,

$$\mathrm{O}(V, q) := \{\text{linear maps } f : V \to V \text{ such that } f^*q = q\}.$$

Since we assumed V is finite-dimensional, the determinant makes sense, so we can also form the **special orthogonal group**

$$\mathrm{SO}(V, q) := \mathrm{O}(V, q) \cap \mathrm{SL}(V).$$

Definition. To a quadratic space (V, q) we associate its **Clifford algebra**

$$\mathcal{C}l(V, q) := T(V)/I_q(V),$$

where $T(V) = \bigoplus_{k=0}^{\infty} T^k(V) = K \oplus V \oplus (V \otimes V) \oplus \cdots$ is the tensor algebra of V, and $I_q(V)$ is the two-sided ideal of $T(V)$ generated by all elements of the form

$$v \otimes v + q(v)1, \qquad v \in V.$$

Remark. The mathematical community is more or less evenly divided when it comes to whether we should set $v^2 = -q(v)1$ or $v^2 = q(v)1$ above. The advantage to inserting the negative sign is that it causes familiar algebras like $\mathbf{C}$ and $\mathbf{H}$ to arise when we apply the construction to positive-definite quadratic spaces. Without the negative sign, these would arise from *negative*-definite quadratic spaces, which people consider weird, for some reason.

Remark. The Clifford algebra is constructed from an "orthogonal geometry" (a vector space equipped with a symmetric bilinear form). There is a similar construction for "symplectic geometries" (vector spaces equipped with skew-symmetric forms), called the *Weyl algebra*.

If $\pi_q : T(V) \to \mathcal{C}l(V, q)$ is the quotient map, we can consider $\pi_q|_V : V \to \mathcal{C}l(V, q)$. One can quickly verify that $I_q(V) \cap V = \{0\}$, so this restriction is injective, allowing us to identify V as a subspace of $\mathcal{C}l(V, q)$.

The algebra $\mathcal{C}l(V, q)$ is generated by $V \subset \mathcal{C}l(V, q)$ subject to the relations $v^2 = -q(v)1$ for $v \in V$. Notice that for $v, w \in V$ we have

$$-q(v + w)1 = (v + w)(v + w) = -q(v)1 - q(w)1 + vw + wv$$

from which we conclude that

$$vw + wv = -2q(v, w),$$

where $2q(v, w) := q(v + w) - q(v) - q(w)$ is the **polarization** of q, i.e. its associated bilinear form.

Universal property and functoriality

Like $T(V)$ and $\Lambda(V)$, the algebra $\mathcal{C}l(V, q)$ is universal (initial) among all linear maps of a certain kind from V into algebras A. More precisely, if A is an algebra, we say a linear map $f : (V, q) \to A$ is **Clifford** (or a **Clifford map**) **with respect to** q if it satisfies

$$f(v)^2 = -q(v)1, \qquad \forall v \in V.$$

Then, using nothing more than the universal property of $T(V)$, the behaviour of quotients, and the uniqueness of initial objects in a category, one shows:

Proposition (universal property of Clifford algebras). If $i = \pi_q|_V : V \hookrightarrow \mathcal{C}l(V, q)$ denotes the inclusion of V into its Clifford algebra as discussed above, then for any algebra A and Clifford map $f : V \to A$, there exists a *unique* algebra homomorphism $\widetilde{f} : \mathcal{C}l(V, q) \to A$ such that $f = \widetilde{f} \circ i$ (that is, $\widetilde{f}$ is an extension of f to the whole Clifford algebra).

$$\begin{array}{ccc}
\mathcal{C}l(V, q) & \xrightarrow{\widetilde{f}} & A \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
V & \xrightarrow{f} & A
\end{array}$$

Furthermore, this property *characterizes* the algebra $\mathcal{C}l(V, q)$ up to isomorphism.

Example. Recall that the exterior algebra $\Lambda(V)$ is characterized by a similar property, involving maps such that $f(v)^2 = 0$ for all $v \in V$. From this, it is clear that in the trivial case when $q \equiv 0$ on V, a map $f : V \to A$ is Clifford with respect to q if and only if $f(v)^2 = 0$ for all $v \in V$, and thus $\mathcal{C}l(V, q) \cong \Lambda(V)$ as algebras. Alternatively, you can just look at the definition and note that when $q \equiv 0$ we are quotienting out by precisely the same ideal used to define $\Lambda(V)$!

Given any morphism $f : (V, q) \to (V', q')$ of quadratic spaces (i.e. f is linear and $f^*q' = q$), then by considering $V' \subset \mathcal{C}l(V', q')$, we can view f as a map $V \to \mathcal{C}l(V', q')$ and invoke the universal property of $\mathcal{C}l(V, q)$ to extend it to a morphism of algebras

$$\mathcal{C}l(f) : \mathcal{C}l(V, q) \to \mathcal{C}l(V', q').$$

Remark. $\mathcal{C}l(-)$ constitutes a functor from the category $\mathbf{QVect}_K$ of quadratic spaces over K, to the category $\mathbf{Alg}_K$ of associative unital algebras over K: its behaviour on objects is $(V,q) \mapsto \mathcal{C}l(V,q)$, and its behaviour on morphisms is

$$\big((V,q) \xrightarrow{f} (V',q')\big) \quad \longmapsto \quad \big(\mathcal{C}l(V,q) \xrightarrow{\mathcal{C}l(f)} \mathcal{C}l(V',q')\big).$$

One simply needs to check that $\mathcal{C}l(f \circ g) = \mathcal{C}l(f) \circ \mathcal{C}l(g)$ and $\mathcal{C}l(\mathrm{id}_V) = \mathrm{id}_{\mathcal{C}l(V,q)}$, both of which are straightforward.

In particular, $\mathrm{O}(V,q)$ extends canonically to a group of automorphisms of $\mathcal{C}l(V,q)$:

$$\mathrm{O}(V,q) \subset \mathrm{Aut}(\mathcal{C}l(V,q)).$$

$\mathbf{Z}/2\mathbf{Z}$-grading and filtration

Definition. We define the **parity automorphism** $\alpha : \mathcal{C}l(V,q) \to \mathcal{C}l(V,q)$ to be the automorphism extending the map $v \mapsto -v$ on V.

Noting that $\alpha^2 = \mathrm{id}$, we obtain a decomposition into ± 1-eigenspaces, called the **even part** and the **odd part** of $\mathcal{C}l(V,q)$ respectively:

$$\mathcal{C}l(V,q) = \underbrace{\mathcal{C}l^0(V,q)}_{+1} \oplus \underbrace{\mathcal{C}l^1(V,q)}_{-1}.$$

Since α is a homomorphism, we note that

$$\mathcal{C}l^i(V,q) \cdot \mathcal{C}l^j(V,q) \subset \mathcal{C}l^{i+j}(V,q)$$

where the indices are taken modulo 2; thus we say $\mathcal{C}l(V,q)$ is a **$\mathbf{Z}/2\mathbf{Z}$-graded algebra** (or, as physicists say, a **superalgebra**). As a consequence, $\mathcal{C}l^0(V,q)$ is a *subalgebra* of $\mathcal{C}l(V,q)$, but $\mathcal{C}l^1(V,q)$ is only a sub*space*.

Remark. Due to "degree reduction" phenomena, $\mathcal{C}l(V,q)$ is *not* a $\mathbf{Z}$-graded algebra like $\Lambda(V)$, or a polynomial ring over a field (in the latter two settings, multiplication by a polynomial *never* drops the degree, but in a Clifford algebra, v^2 is a scalar for $v \in V$). However, it does inherit the structure of a **filtered algebra** from $T(V)$. Indeed, $T(V)$ has a natural filtration $\widetilde{\mathcal{F}}^0 \subset \widetilde{\mathcal{F}}^1 \subset \cdots \subset T(V)$, where

$$\widetilde{\mathcal{F}}^r := \bigoplus_{s \leq r} T^s(V).$$

Setting $\mathcal{F}^r := \pi_q(\widetilde{\mathcal{F}}^r)$, we see that $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{C}l(V,q)$ is a filtration of the Clifford algebra.

We record two results here for interest; they will be restated with proof when we need them.

Theorem. The associated graded algebra of $\mathcal{C}l(V,q)$,

$$\mathcal{G}^* = \bigoplus_{r \geq 0} \mathcal{G}^r, \qquad \mathcal{G}^r := \mathcal{F}^r / \mathcal{F}^{r-1},$$

is isomorphic to $\Lambda(V)$.

Theorem. $\mathcal{C}l(V,q)$ is naturally isomorphic, as a vector space, to $\Lambda(V)$.

There is a second fundamental involution on the algebra $\mathcal{C}l(V,q)$. The tensor algebra $T(V)$ has an involution given on pure tensors by reversal of order:

$$v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes \cdots \otimes v_1.$$

This map clearly preserves the ideal $I_q(V)$, so it descends to a map

$$(\quad)^t : \mathcal{C}l(V,q) \to \mathcal{C}l(V,q)$$

which we call the **transpose**. Note that it is an **antiautomorphism**, that is, $(\varphi\psi)^t = \psi^t \varphi^t$.

Pin and Spin groups

Since $\mathcal{C}l(V,q)$ is an algebra, we can consider its multiplicative group of invertible elements, defined to be

$$\mathcal{C}l^\times(V,q) := \{\varphi \in \mathcal{C}l(V,q) : \exists \varphi^{-1} \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\},$$

which contains all $v \in V$ with $q(v) \neq 0$ (the so-called **non-null** vectors), as is easily seen from the relation $v^2 = -q(v)1$. This is a Lie group of dimension 2^n, where $n = \dim V < \infty$, so there is an associated Lie algebra $\mathfrak{cl}^\times(V,q) = \mathcal{C}l(V,q)$ with Lie bracket given by

$$[x,y] = xy - yx.$$

We have the usual **adjoint representation** of a Lie group on its Lie algebra,

$$\mathrm{Ad} : \mathcal{C}l^\times(V,q) \to \mathrm{Aut}(\mathcal{C}l(V,q)), \qquad \mathrm{Ad}_\varphi(x) = \varphi x \varphi^{-1}.$$

Taking the derivative/pushforward/differential of this, we get the morphism of Lie algebras

$$\mathrm{ad} : \mathfrak{cl}^\times(V,q) \to \mathrm{Der}(\mathcal{C}l(V,q)), \qquad \mathrm{ad}_y(x) = [y,x].$$

Since $\mathcal{C}l(V,q)$ is finite-dimensional, there is a natural exponential mapping $\exp : \mathfrak{cl}^\times(V,q) \to \mathcal{C}l^\times(V,q)$, defined as usual by

$$\exp(y) = \sum_{m=0}^{\infty} \frac{y^m}{m!}.$$

We now make an important observation about the adjoint representation, which follows from a simple calculation.

Proposition. Let $v \in V$ satisfy $q(v) \neq 0$. Then for all $w \in V$,

$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v \tag{*}$$

hence, in particular, $\mathrm{Ad}_v(V) = V$.

Proof. Noting that $v^{-1} = -v/q(v)$, we have, simply using the fact that $vw + wv = -2q(v,w)$, that

$$-q(v)\mathrm{Ad}_v(w) = -q(v)vwv^{-1} = vwv = -v^2w - 2q(v,w)v = q(v)w - 2q(v,w)v. \qquad \square$$

Hence we are lead naturally to consider the subgroup of those $\varphi \in \mathcal{C}l^\times(V,q)$ such that $\mathrm{Ad}_\varphi(V) = V$. The above shows that this subgroup contains all non-null $v \in V$.

Definition. $\mathrm{P}(V,q)$ is defined to be the subgroup of $\mathcal{C}l^\times(V,q)$ generated by all elements $v \in V$ with $q(v) \neq 0$.

Since the above also tells us that $\mathrm{Ad}_v^* q = q$ for all $v \in V$ with $q(v) \neq 0$, there is a representation

$$\mathrm{P}(V,q) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V,q) := \{\lambda \in \mathrm{GL}(V) : \lambda^* q = q\}.$$

The group $\mathrm{P}(V,q)$ has some important subgroups.

Definition. The **Pin group** of (V,q) is the subgroup $\mathrm{Pin}(V,q)$ of $\mathrm{P}(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$, that is, the "generalized unit sphere" of (V,q). The associated **Spin group** of (V,q) is defined by

$$\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap \mathcal{C}l^0(V,q).$$

Notice that the RHS of (*) is just the linear map $\rho_v : V \to V$ given by reflection across the hyperplane $v^\perp := \{u \in V : q(u,v) = 0\}$. Unfortunately, there is a pesky negative sign on the LHS of (*). This implies in particular that if $\dim V$ is odd, Ad_v is always orientation preserving. This is really bad, since after all, we are trying to get a surjective map to $\mathrm{O}(V,q)$. To fix this, we just modify Ad a bit, using the parity automorphism α.

Definition. We define the **twisted adjoint representation** by

$$\widetilde{\mathrm{Ad}} : \mathcal{C}l^\times(V,q) \to \mathrm{GL}(\mathcal{C}l(V,q)), \qquad \widetilde{\mathrm{Ad}}_\varphi(y) = \alpha(\varphi) y \varphi^{-1}.$$

Clearly, $\widetilde{\mathrm{Ad}}_{\varphi_1\varphi_2} = \widetilde{\mathrm{Ad}}_{\varphi_1} \circ \widetilde{\mathrm{Ad}}_{\varphi_2}$ and $\widetilde{\mathrm{Ad}}_\varphi = \mathrm{Ad}_\varphi$ for even elements φ. Also, (*) tells us that

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{q(v,w)}{q(v)} v.$$

Definition. We define the **Clifford group** $\widetilde{\mathrm{P}}(V,q)$ to be the subgroup of all $\varphi \in \mathcal{C}l^\times(V,q)$ whose action under the twisted adjoint representation preserves V. In symbols,

$$\widetilde{\mathrm{P}}(V,q) := \{\varphi \in \mathcal{C}l^\times(V,q) : \widetilde{\mathrm{Ad}}_\varphi(V) = V\}.$$

Clearly, $\mathrm{P}(V,q) \subset \widetilde{\mathrm{P}}(V,q)$. Furthermore, we have the following proposition, which shows how crucial it is we use the *twisted* adjoint representation.

Proposition. Suppose $\dim V < \infty$ and q is nondegenerate. Then the kernel of the homomorphism

$$\widetilde{\mathrm{P}}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{GL}(V)$$

is exactly the group $K^\times$ of nonzero multiples of 1.

Proof. Choose a basis $\{v_1,\dots,v_n\}$ for V such that $q(v_i) \neq 0$ for all i and $q(v_i,v_j) = 0$ for all $i \neq j$. Suppose $\varphi \in \mathcal{C}l^\times(V,q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$, that is, suppose φ has the property that $\alpha(\varphi)v = v\varphi$ for all $v \in V$. Write $\varphi = \varphi_0 + \varphi_1$, where φ_0 is even and φ_1 is odd, and observe that

$$\begin{aligned} v\varphi_0 &= \varphi_0 v \\ -v\varphi_1 &= \varphi_1 v \end{aligned}$$

for all $v \in V$. The terms φ_0 and φ_1 can be written as polynomial expressions in $v_1,\dots,v_n$. Successive use of the fact that $v_iv_j = -v_jv_i - 2q(v_i,v_j)$ shows that φ_0 can be expressed as $\varphi_0 = a_0 + v_1a_1$ where a_0 and a_1 are polynomial expressions in $v_2,\dots,v_n$. Applying α shows that a_0 is even and a_1 is odd. Setting $v = v_1$ in the centered equation above, we see that

$$v_1a_0 + v_1^2a_1 = a_0v_1 + v_1a_1v_1 = v_1a_0 - v_1^2a_1.$$

Hence, $v_1^2a_1 = -q(v_1)a_1 = 0$, and so $a_1 = 0$. This implies that φ_0 does not involve v_1. Proceeding inductively, we see that φ_0 does not involve any of the terms $v_1,\dots,v_n$ and so $\varphi_0 = t\cdot 1$ for $t \in K$.

The analogous argument can now be applied to φ_1. Write $\varphi_1 = a_1 + v_1a_0$, where a_0 and a_1 do not involve v_1. Note that a_1 is odd and a_0 is even; and therefore from the centered equations above,

$$-v_1a_1 - v_1^2a_0 = a_1v_1 + v_1a_0v_1 = -v_1a_1 + v_1^2a_0.$$

Hence, $a_0 = 0$ and so φ_1 is independent of v_1. By induction, φ_1 is independent of $v_1,\dots,v_n$ and so $\varphi_1 = 0$. Now we have $\varphi = \varphi_0 + \varphi_1 = t\cdot 1 \in K$. But $\varphi \neq 0$, so $\varphi \in K^\times$. □

Remark. This is false if we do not assume q to be nondegenerate; consider $\mathcal{C}l(V,0) = \Lambda(V)$:

$$\alpha(1+v_1v_2)v(1+v_1v_2)^{-1} = (1+v_1v_2)v(1-v_1v_2) = v.$$

Hence the kernel includes many non-scalar elements.

Definition. Introduce the **norm** mapping $N : \mathcal{C}l(V,q) \to \mathcal{C}l(V,q)$ defined by setting

$$N(\varphi) = \varphi \cdot \alpha(\varphi^t).$$

It is easy to see that $\alpha(\varphi^t) = (\alpha(\varphi))^t$. Note that $N(v) = q(v)$ for $v \in V$.

The importance of the norm is evident from the following proposition.

Proposition. Suppose that $\dim V < \infty$ and q is nondegenerate. Then the restriction of N to the group $\widetilde{\mathrm{P}}(V,q)$ gives a homomorphism

$$N : \widetilde{\mathrm{P}}(V,q) \to K^\times$$

into the multiplicative group of nonzero multiples of the identity in $\mathcal{C}l(V,q)$.

Proof. □

Continuing to assume $\dim V < \infty$ and q is nondegenerate, we get:

Corollary. The transformations $\widetilde{\mathrm{Ad}}_\varphi : V \to V$ for $\varphi \in \widetilde{\mathrm{P}}(V,q)$ preserve the quadratic form q. Hence, there is a homomorphism

$$\widetilde{\mathrm{Ad}} : \widetilde{\mathrm{P}}(V,q) \to \mathrm{O}(V,q).$$

Proof. □

Now, returning to the group $\mathrm{P}(V,q) \subset \widetilde{\mathrm{P}}(V,q)$, we observe that by definition,

$$\mathrm{P}(V,q) = \{v_1 \cdots v_r \in \mathcal{C}l(V,q) : v_1, \ldots, v_r \text{ is a finite sequence from } V^\times\}.$$

Recall the twisted adjoint representation gives a homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{P}(V,q) \to \mathrm{O}(V,q)$ such that

$$\widetilde{\mathrm{Ad}}_{v_1 \cdots v_r} = \rho_{v_1} \circ \cdots \circ \rho_{v_r}, \qquad \text{where} \qquad \rho_v(w) = w - 2\frac{q(w,v)}{q(v)}v \tag{**}$$

that is, ρ_v is reflection across $v^\perp$. Thus the image of $\mathrm{P}(V,q)$ under $\widetilde{\mathrm{Ad}}$ is exactly the group generated by reflections. The following classical result tells us that this is always the whole orthogonal group.

Theorem (Cartan-Dieudonné). Let q be a nondegenerate quadratic form on a finite-dimensional vector space V. Then every element $g \in \mathrm{O}(V,q)$ can be written as a product of r reflections

$$g = \rho_{v_1} \circ \cdots \circ \rho_{v_r}$$

where $r \leq \dim(V)$.

Proof. Omitted. □

This theorem tells us that the homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{P}(V,q) \to \mathrm{O}(V,q)$ is surjective. Furthermore, we could consider the group $\mathrm{SP}(V,q) = \mathrm{P}(V,q) \cap \mathcal{C}l^0(V,q)$ and, since $\dim V < \infty$, the special orthogonal group $\mathrm{SO}(V,q)$ we defined previously. The same theorem (C-D) also says that:

Corollary. $\widetilde{\mathrm{Ad}} : \mathrm{SP}(V,q) \to \mathrm{SO}(V,q)$ is surjective.

Proof. ... □

Thus from Cartan-Dieudonné we conclude that

$$\mathrm{SO}(V,q) = \{\rho_{v_1} \circ \cdots \circ \rho_{v_r} : q(v_j) \neq 0 \text{ and } r \text{ is even}\}.$$

From the definition (observation about $\mathrm{P}(V,q)$ above) we see that

$$\mathrm{SP}(V,q) = \{v_1 \cdots v_r \in \mathrm{P}(V,q) : r \text{ is even}\}.$$

The surjectivity of $\widetilde{\mathrm{Ad}} : \mathrm{SP}(V,q) \to \mathrm{SO}(V,q)$ follows immediately by (**).

... Here is the main result.

Theorem (main result). Let V be a finite-dimensional vector space over a spin field K, and suppose q is a nondegenerate quadratic form on V. Then there are short exact sequences

$$0 \to F \to \mathrm{Spin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,q) \to 1$$

$$0 \to F \to \mathrm{Pin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q) \to 1$$

where

$$F = \begin{cases} \mathbf{Z}/2\mathbf{Z} = \{1,-1\} & \text{if } \sqrt{-1} \notin K \\ \mathbf{Z}/4\mathbf{Z} = \{\pm 1, \pm\sqrt{-1}\} & \text{otherwise.} \end{cases}$$

These sequences hold for general fields provided that $\mathrm{SO}(V,q)$ and $\mathrm{O}(V,q)$ are replaced by appropriate normal subgroups of $\mathrm{O}(V,q)$.