HOPF ALGEBRAS, SYMMETRIC FUNCTIONS AND REPRESENTATIONS

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1. Introduction

There are two primary goals in this talk. We first discuss the notion of a Hopf algebra and its relation to representation theory. In particular, we shall see how the rich structure of Hopf algebras unifies various representation theoretic constructions. Second, we will examine how Hopf algebras come up in combinatorics by studying the ring of symmetric functions.

1.1. **Motivation.** Much of representation theory can be unified by considering the representation theory of associative algebras. Specifically, the representation theory of Lie algebras may be studied via the representations of universal enveloping algebras; the representation theory of finite groups studied via the representation theory of the group algebra; the representation theory of quivers studied through the representation theory of the path algebra; and so forth. The idea is that the representation theoretic aspects of a particular mathematical object is captured by some associative algebra that is, in a rather precise way, naturally related to the original object.

The associative algebras arising in each of the above constructions, however, are much more structured and well-behaved than general associative algebras due to the representation theory underlying the original objects. To illustrate more precisely what we mean here, let A be any algebra over $\mathbf{C}$—for convenience, we will work over the complex numbers, but for the most part, the base field will not be of concern—and consider two representations V and W of A. There are general constructions which we would like to perform on V and W and obtain new representations of A, namely, we would like to make sense of the sum representation $V \oplus W$, the tensor representation $V \otimes W$ and the dual representation V^*.

The sum representation $V \oplus W$ is easy to make sense of even for general algebras A: for $v \oplus w \in V \oplus W$, we can define an A-action by acting on each component separately, $a \cdot (v \oplus w) = (a \cdot v) \oplus (a \cdot w)$. It is not hard to check that this actually defines a representation on $V \oplus W$.

The tensor and dual representations, however, are a bit more tricky. The issue with the tensor product, for instance, is that $V \otimes W$ has an induced $A \otimes A$-action from the A-action on each of the components, but this does not automatically yield an A-action on $V \otimes W$. The naïve thing to do here is to attempt to define an A-action simply as above, by having A act on the tensor component-wise. This happens to work in the case where $A = \mathbf{C}[G]$ is a group algebra, but this does not work, say, when A is the universal enveloping algebra of a Lie algebra, as we have seen.

One solution to this problem is to fix a map $\Delta : A \to A \otimes A$, called **comultiplication**, and have A act on $V \otimes W$ through the map Δ. That is, we define an A-action on $V \otimes W$ by

$$a \cdot (v \otimes w) \coloneqq \Delta(a) \cdot (v \otimes w),$$

where the action on the right hand side is the $A \otimes A$-action inherited from the original actions. Of course, not just any map Δ will do. We require that Δ is at least an algebra homomorphism so that the resulting action respects the associative and unit laws demanded of actions. But we require a bit more: tensor products are associative, so we must have that Δ satisfy some additional properties to reflect that fact. These will be encapsulated by what are known as the coassociative and counitary laws, which we shall detail shortly.

For instance, let G be a finite group and let $A \coloneqq k[G]$ be the group algebra. For any $g \in G$, set $\Delta(g) \coloneqq g \otimes g$ and extend Δ linearly to the rest of A. Then we have seen that if V and W are representations of $k[G]$, then

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$V \otimes W$ may be turned into a representation of $k[G]$ by defining

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w) = \Delta(g) \cdot (v \otimes w).$$

Now let $\mathfrak{g}$ be a Lie algebra and let $A := U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For any $X \in U(\mathfrak{g})$, set $\Delta(X) := X \otimes 1 + 1 \otimes X$. For representations V and W of $U(\mathfrak{g})$, $V \otimes W$ may be turned into a representation of $U(\mathfrak{g})$ by defining

$$X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w) = \Delta(X) \cdot (v \otimes w).$$

This is one of the ways in which both the group algebra and the universal enveloping algebra carry additional algebraic structure which, in some sense, encodes the essential representation theoretic constructions we care about.

1.2. **Bialgebras.** To make precise the ideas of the preceding discussion, it is useful to recall the definition of an algebra. Let A be a k-vector space. To make A into a k-algebra, we need a linear map $\mu : A \otimes A \to A$, the multiplication of A, and a linear map $\eta : k \to A$, the unit of A, such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes \mathrm{id}"] \arrow[dr, dash, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, dash, "\cong"] \\ & A & \end{tikzcd}$$

where the diagram on the left expresses associativity of multiplication and the diagram on the right essentially says that η maps the 1 of k to the multiplicative unit of A. If we take the categorical dual of the diagrams above, we obtain the definition of a coalgebra. Specifically, a k-coalgebra structure on A is given by two linear maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\mathrm{id}\otimes\Delta"] \\ A \otimes A \arrow[r, "\Delta\otimes \mathrm{id}"] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, dash, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, dash, "\cong"] & \\ k \otimes A & A \otimes A \arrow[l, "\epsilon\otimes \mathrm{id}"] \arrow[r, "\mathrm{id}\otimes\epsilon"'] & A \otimes k \end{tikzcd}$$

where the diagram on the left is called the **coassociative law** and the diagram on the right is the **counitary law**. The coassociative law expresses the reasonable demand that, when applying the coproduct multiple times, it should not matter which factor of the tensor product we make the later applications to. The counitary law essentially says that the coproduct is symmetric, as measured by the counit. Note that the counit also captures the trivial representation for the algebra.

A k-vector space A along with maps $\mu, \eta, \Delta, \epsilon$ is said to be a **bialgebra** if (A, μ, η) is a k-algebra and (A, Δ, ϵ) is a coalgebra such that Δ and ϵ are k-algebra homomorphisms. One may show that Δ and ϵ are k-algebra homomorphisms if and only if μ and η are k-coalgebra homomorphisms, so the apparently asymmetric definition is but an illusion.

1.3. **Commutativity and Cocommutativity.** We digress from our representation theoretic considerations and discuss a property of bialgebras. Let B be a bialgebra. We are very familiar with the notion of commutativity: for every $a, b \in B$, $\mu(a \otimes b) = \mu(b \otimes a)$. To formulate the analogous property for the comultiplication, it will be useful to encode this condition into a diagram. Let $T : B \otimes B \to B \otimes B$ be the *twist* map given by $a \otimes b \mapsto b \otimes a$. Then the commutativity property is equivalent to

$$\begin{tikzcd} B \otimes B \arrow[rr, "T"] \arrow[dr, "\mu"'] & & B \otimes B \arrow[dl, "\mu"] \\ & B & \end{tikzcd}$$

being a commutative diagram. Reversing the arrows and replacing multiplication μ by comulitplication Δ, we say that a bialgebra B is **cocommutative** if the following diagram

$$\begin{tikzcd} & B \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ B \otimes B \arrow[rr, "T"'] & & B \otimes B \end{tikzcd}$$

commutes. Algebraically, for $b \in B$, if we write $\Delta(b) = \sum_i b_i' \otimes b_i''$, then B is cocommutative if for all $b \in B$,

$$\sum_i b_i' \otimes b_i'' = \sum_i b_i'' \otimes b_i'.$$

Thus cocommutativity is a symmetry condition on the comultiplication.

1.4. **Hopf Algebras.** As per the previous section, we can make sense of the direct sum representation with just an algebra structure and we can make sense of the tensor product representation with the addition of a coalgebra structure. Thus, bialgebras are structures in which we have a natural way to make sense of both direct sum and tensor product representations. The definition of a Hopf algebra completes the picture by allowing us to make general sense of the dual representation.

A **Hopf algebra** is a bialgebra A along with an algebra homomorphism $S : A \to A^{op}$, where A^{op} denotes the opposite algebra—the algebra with the same underlying vector space but where multiplication is flipped, $a \cdot_{op} b = ba$—such that when V^* is endowed with the A-action

$$(a \cdot f)(x) := f(S(a) \cdot x),$$

then the evaluation maps $V^* \otimes V \to k$ and $V \otimes V^* \to k$ are A-linear maps. To see what this means, let $f \in V^*$, $v \in V$ and $a \in A$ and let $ev : V^* \otimes V \to k$ be the evaluation map. Linearity means that $ev(a \cdot (f \otimes v)) = a \cdot ev(f \otimes v)$, so let us compute both sides. Write $\Delta(a) = \sum_i a_i' \otimes a_i''$. The left hand side is

$$ev(a \cdot (f \otimes v)) = ev\left(\sum_i (a_i' \cdot f) \otimes (a_i'' \cdot v)\right) = f\left(\sum_i S(a_i')a_i'' \cdot v\right).$$

The right hand side is

$$a \cdot ev(f \otimes v) = \epsilon(a)f(v) = f(\epsilon(a)v).$$

For the equality $ev(a \cdot (f \otimes v)) = a \cdot ev(f \otimes v)$ for all f and v, we must have

$$\sum_i S(a_i')a_i'' = \epsilon(a) \cdot 1 = \eta(\epsilon(a)).$$

A similar computation for the other evaluation map $V \otimes V' \to k$ shows

$$\sum_i a_i' S(a_i'') = \epsilon(a) \cdot 1 = \eta(\epsilon(a)).$$

These calculations show that the algebraic condition that captures the dual representation construction is that $S : A \to A^{op}$ satisfies $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon$ and $\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon$, which may be expressed in a diagram as

$$\begin{tikzcd} & A \otimes A \arrow[rr, "S \otimes \mathrm{id}"] & & A \otimes A \arrow[dr, "\mu"] & \\ A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & A \\ & A \otimes A \arrow[rr, "\mathrm{id} \otimes S"'] & & A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

Or equivalently, if $\Delta(a) = \sum_i a_i' \otimes a_i''$,

$$\sum_i S(a_i')a_i'' = \eta(\epsilon(a)) = \sum_i a_i' S(a_i'').$$

Yet another way to understand the antipode is through the induced k-algebra structure on $\mathrm{Hom}(A, A)$ for A a bialgebra. Specifically, define a product on the k-vector space $\mathrm{Hom}(A, A)$ by setting, for $f, g \in \mathrm{Hom}(A, A)$,

$$f * g := \mu \circ (f \otimes g) \circ \Delta.$$

One can check that this product makes $\mathrm{Hom}(A, A)$ into a k-algebra with unit $\eta \circ \epsilon$. Then the antipode S can be characterized as the inverse to the identity map id_A in this algebra. A consequence of this characterization is that if an antipode exists on a bialgebra, then it must be unique.

A few remarks before we move on. First, one intuition for the antipode is that it should be a sort of replacement for inverting an element in a group. One can show that the antipode on the group algebra $k[G]$ acts on basis elements g by $g \mapsto g^{-1}$. Another way to think about the antipode, one which is perhaps more helpful combinatorially, is that it somehow captures a sort of shuffling or mixing of elements. Indeed, the antipode is the inverse to the identity map in the convolution algebra, so in some way, it should convolve the underlying objects in some well-behaved way.

1.5. **Graded and Connected.** A Hopf algebra H is said to be **graded** if there exists linear subspaces H_n of H, $n \geqslant 0$, such that $H = \bigoplus_{n \geqslant 0} H_n$ and the multiplication and comultiplication respect the grading in the sense that

$$\mu(H_i \otimes H_j) \subseteq H_{i+j}, \quad \Delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$$

and that $\eta(k) \subseteq H_0$. A Hopf algebra is said to be **connected** if $H_0 = k$. Note that the terminology for connectedness comes from topology: one of the original examples of Hopf algebras was the cohomology ring associated with topological spaces and the condition that the zeroth cohomology group is the base field is equivalent to the fact that the topological space is connected. Let us now state two properties of antipodes which we shall refer to.

Proposition 1. If H is a graded and connected bialgebra, then H admits a unique graded linear map $S : H \to H$ making H into a Hopf algebra.

Proposition 2. If H is a commutative or cocommutative Hopf algebra with antipode S, then $S^2 = \mathrm{id}_H$.

Proposition 3. Let H be a Hopf algebra with antipode S, then for $x \in H$ with $\Delta(x) = 1 \otimes x + x \otimes 1$, $S(x) = -x$.

1.6. **Hopf Algebras and Combinatorics.** Hopf algebra techniques in combinatorics were pioneered by Gian-Carlo Rota in his work to provide firm foundations for combinatorics. Rota realized that the algebraic structure on Hopf algebras naturally encodes the notions of assembling and disassembling objects. Specifically, the multiplication of the algebra structure allows one to put smaller pieces together to obtain some composite object whereas the comultiplication of the coalgebra is a way to break a complicated object to simpler constituents. The antipode then comes into the picture usually as a distinguished mapping among the combinatorial objects being considered. In particular, most Hopf algebras associated to combinatorial objects will be either commutative or cocommutative, so Proposition 2 implies that the associated antipode is a distinguished involution.

1.7. **Example: Group Algebra.** Let G be a group and let $H := k[G]$ be the group algebra of G. We have seen that H has the structure of a bialgebra with the multiplication a linear extension of the group multiplication, the coproduct given by $g \mapsto g \otimes g$, the unit given by $1 \mapsto e$ and the counit given by $g \mapsto \delta_{eg}$, where $e \in G$ is the group identity. Also, we have remarked that the antipode of H is given by the linear extension of the mapping $g \mapsto g^{-1}$.

1.8. **Example: Universal Enveloping Algebra.** Let $H := U(\mathfrak{g})$ be the universal enveloping algebra of some Lie algebra $\mathfrak{g}$. The multiplication on H is the product of $U(\mathfrak{g})$, the unit maps $1 \mapsto e$, the comultiplication is given by $X \mapsto X \otimes 1 + 1 \otimes X$ and the counit is given by $X \mapsto \delta_{eX}$. The antipode S on H is given by $X \mapsto -X$.

1.9. **Example: Polynomial Algebra.** One example that is more combinatorial in flavour is given by the polynomial bialgebra. Let $H = k[x]$ be the vector space of polynomials in x over the field k. It is elementary that H is an algebra. To define the comultiplication, let $n \geqslant 1$ and consider the mapping

$$\Delta(x^n) := \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

One may check that the linear extension of this map indeed defines a comultiplication on H. The counit of H is given by $x^n \mapsto \delta_{0n}$. Finally, the antipode of H is given by $x \mapsto -x$.

2. Symmetric Functions

The ring of symmetric functions is perhaps one of the most beautiful and well-studied mathematical objects. Not only are symmetric functions fascinating on their own, but their properties find applications in other areas of mathematics, including algebra, representation theory, geometry and combinatorics. Here, we shall review some of the basic properties of symmetric functions, with the perspective that the symmetric functions have the structure of a Hopf algebra.

2.1. **Bialgebra of Symmetric Functions.** Let k be a field, let $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$ be an infinite set of variables and let $k[\![\mathbf{x}]\!]$ be the ring of formal power series in the variables $\mathbf{x}$. Let $n \in \mathbf{N}$, then by a **composition** α of n, we mean a sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ for which each $\alpha_i \geqslant 0$ and $\sum_i \alpha_i = n$. For any composition α of n, we write $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots$ and say that $\mathbf{x}^\alpha$ is a monomial of degree n.

For each $n \geqslant 1$, denote by $\mathfrak{S}_n$ the symmetric group on n letters. We can view $\mathfrak{S}_n$ as a subgroup of $\mathfrak{S}_m$ for all $m \geqslant n$ by thinking of $\pi \in \mathfrak{S}_n$ as acting on the first n letters and leaving the last $m-n$ letters fixed. That is, we view $\pi \in \mathfrak{S}_n$ as the element $\pi' \in \mathfrak{S}_m$ for which $\pi'(i) = \pi(i)$ for $i \in [n]$ and $\pi'(i) = i$ for $i \in [m] - [n]$. In this way, we have an ascending chain of group $\mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \mathfrak{S}_3 \dots$ and so we can construct the infinite symmetric group as $\mathfrak{S}_\infty := \bigcup_{n=1}^\infty \mathfrak{S}_n$: these are permutations that permute finitely many positive integers and leave all others fixed.

There is a natural action of $\mathfrak{S}_\infty$ on $k[\![\mathbf{x}]\!]$ given by permuting the variables. Specifically, for $\pi \in \mathfrak{S}_\infty$ and $f(\mathbf{x}) \in k[\![\mathbf{x}]\!]$ by

$$\pi \cdot f(x_1, x_2, x_3, \dots) := f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \dots).$$

The **ring of symmetric functions** $\Lambda = \Lambda(\mathbf{x}) = k[\![\mathbf{x}]\!]^{\mathfrak{S}_\infty}$ is the $\mathfrak{S}_\infty$-invariant subalgebra of $k[\![\mathbf{x}]\!]$. That is to say that Λ consists of all the power series for which permuting any of the variables leaves the power series the same. Notice that there is a natural grading on Λ as

$$\Lambda = \bigoplus_{n \geqslant 0} \Lambda_n$$

where Λ_n consists of all symmetric functions which are **homogeneous of degree** n, which is to say that each monomial appearing has degree n. Also notice that Λ is connected, since $\Lambda_0 = k$. This tells us that once we are able to define a bialgebra structure on Λ, then Λ can be turned into a Hopf algebra.

To obtain a coproduct on Λ, consider splitting the variables into two sets $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \dots, y_1, y_2, \dots)$, then we have a ring homomorphism

$$\begin{aligned} k[\![\mathbf{x}]\!] \otimes k[\![\mathbf{y}]\!] &\to k[\![\mathbf{x}, \mathbf{y}]\!] \\ f(\mathbf{x}) \otimes g(\mathbf{y}) &\mapsto f(\mathbf{x}) g(\mathbf{y}). \end{aligned}$$

One can either show directly, or else it shall become apparent when we discuss bases for Λ, that the above map restricts to an isomorphism

$$\Lambda(\mathbf{x}, \mathbf{y}) = k[\![\mathbf{x}]\!]^{\mathfrak{S}_\infty} \otimes k[\![\mathbf{y}]\!]^{\mathfrak{S}_\infty} \cong k[\![\mathbf{x}, \mathbf{y}]\!]^{\mathfrak{S}_\infty \times \mathfrak{S}_\infty},$$

where $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$ is viewed as the subgroup of $\mathfrak{S}_\infty$ that permutes the variables $\mathbf{x}$ and $\mathbf{y}$ separately. Observe that if $f(\mathbf{x}, \mathbf{y})$ is invariant under permutation of all the variables $\mathbf{x}$ and $\mathbf{y}$ together, then $f(\mathbf{x}, \mathbf{y})$ is also invariant under permuting the variables $\mathbf{x}$ amongst themselves and permuting the variables $\mathbf{y}$ amongst themselves. In other words, we have a inclusion

$$\Lambda(\mathbf{x}, \mathbf{y}) = k[\![\mathbf{x}, \mathbf{y}]\!]^{\mathfrak{S}_\infty} \hookrightarrow k[\![\mathbf{x}, \mathbf{y}]\!]^{\mathfrak{S}_\infty \times \mathfrak{S}_\infty} \cong \Lambda \times \Lambda.$$

Hence we obtain a comultiplication

$$\begin{aligned} \Delta : \Lambda(\mathbf{x}) &\to \Lambda(\mathbf{x}, \mathbf{y}) \hookrightarrow \Lambda \otimes \Lambda \\ f(\mathbf{x}) &\mapsto f(\mathbf{x}, \mathbf{y}). \end{aligned}$$

In other words, the coproduct of a symmetric function $f(\mathbf{x})$ is the power series $f(\mathbf{x}, \mathbf{y})$ which is symmetric in both $\mathbf{x}$ and $\mathbf{y}$ independently, i.e. $f(\mathbf{x}, \mathbf{y})$ is fixed under permuting the variables $\mathbf{x}$ amongst themselves and permuting the variables $\mathbf{y}$ amongst themselves.

2.2. **Bases for Symmetric Functions.** The structure of Λ is made much clearer by examining generators for Λ. There are various generators for Λ, each with their own advantages. The simplest basis Λ is given by **monomial symmetric functions**

$$m_\lambda(\mathbf{x}) \coloneqq \sum_{\alpha \in \mathfrak{S}_n \cdot \lambda} \mathbf{x}^\alpha$$

where $\lambda : (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0)$ is a partition and the action of $\mathfrak{S}_\infty$ on a partition λ is given by permuting the entries if the partition. Each $m_\lambda(\mathbf{x})$ is the symmetrization of the monomial $\mathbf{x}^\lambda$ and so it is clear that the collection of all monomial symmetric functions yields a k-basis for Λ.

From the monomial symmetric functions, we can define three other families of symmetric functions. For each $n \geqslant 1$, define the **power sum symmetric functions** p_n, the **elementary symmetric functions** e_n and the **complete homogeneous symmetric functions** h_n by

$$p_n \coloneqq m_{(n)} = \sum_{i \geqslant 1} x_i^n, \qquad e_n \coloneqq m_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}, \qquad h_n \coloneqq \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leqslant \cdots \leqslant i_n} x_{i_1} \dots x_{i_n}$$

For a partition $\lambda : (\lambda_1 \geqslant \cdots \geqslant \lambda_\ell > 0)$, write

$$p_\lambda \coloneqq p_{\lambda_1} \dots p_{\lambda_\ell}, \qquad e_\lambda \coloneqq e_{\lambda_1} \dots e_{\lambda_\ell}, \qquad h_\lambda \coloneqq h_{\lambda_1} \dots h_{\lambda_\ell}.$$

Since $\{ m_\lambda \}_\lambda$ is a linear basis for Λ, each of the above symmetric functions may be expressed as a linear combination of the m_λ. By looking at the expansion of p_λ, e_λ and h_λ in terms of the m_λ and putting those coefficients in a matrix indexed by partitions in lexicographic order, one can show that such coefficient matrices are always upper triangular. In particular, this implies that each of the collections $\{ p_\lambda \}_\lambda$, $\{ e_\lambda \}_\lambda$ and $\{ h_\lambda \}_\lambda$ are bases for Λ. Notice that this means that the $\{ p_n \}_n$, $\{ h_n \}_n$ and $\{ e_n \}_n$ actually generate Λ as an algebra.

With these bases in hand, let us understand the action of the coproduct a little bit better. The coproduct has the following effect on the bases discussed so far:

(1) $\Delta(m_\lambda) = \sum_{\mu \sqcup \nu = \lambda} m_\mu \otimes m_\nu$;
(2) $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$;
(3) $\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$;
(4) $\Delta(h_n) = \sum_{i+j=n} h_i \otimes h_j$,

where $\mu \sqcup \nu$ is the partition obtained by taking the disjoint union of the partitions as multisets and then one reorders the result to obtain a partition. To see why the above formulae are true, one simply needs to recall the definition of the coproduct and stare at the symbols for a while.

2.3. **Antipode on Symmetric Functions.** Now that we have a clearer understanding of the bialgebra structure on Λ, we can attempt to find the antipode on Λ. Recall that since Λ is graded and connected, Proposition 2 implies that Λ does indeed have an antipode S. We immediately note that since $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$ that $S(p_n) = -p_n$. We now claim that for each $n \geqslant 1$,

$$S(e_n) = (-1)^n h_n, \quad S(h_n) = (-1)^n e_n.$$

To see why this might be true, consider the generating functions

$$H(t) \coloneqq \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \sum_{n \geqslant 0} h_n(\mathbf{x}) t^n, \qquad E(t) \coloneqq \prod_{i=1}^{\infty} (1 + x_i t) = \sum_{n \geqslant 0} e_n(\mathbf{x}) t^n,$$

where the first is true by the same reasoning used to obtain the generating function for partitions, and the second is true by noting that the coefficient of t^n consists of all the distinct ways to pick n of the variables x_i. From these generating functions, we have that $E(-t)H(t) = 1$ and from which comparing coefficients yields

$$\sum_{i+j=n} (-1)^i e_i h_j = \delta_{0n}$$

Since Λ is a graded connected Hopf algebra, $\eta(\epsilon(e_n)) = \delta_{0n} = \eta(\epsilon(h_n))$ and so the defining equations for the antipode read

$$\sum_{i+j=n} S(e_i)e_j = \delta_{0n} = \sum_{i+j=n} e_i S(e_j), \qquad \sum_{i+j=n} S(h_i)h_j = \delta_{0n} = \sum_{i+j=n} h_i S(h_j).$$

Comparing these equations to the equalities from the generating functions, we see that $S(e_n) = (-1)^n h_n$ and $S(h_n) = (-1)^n e_n$.

Let us note one additional thing. The identities resulting from the generating functions may be rearranged to obtain the Newton identities:

$$\begin{aligned}
e_0 &= h_0 = 1, \\
e_n &= e_{n-1}h_1 - e_{n-2}h_2 + e_{n-3}h_3 - \cdots + (-1)^{n-1}h_n, \\
h_n &= h_{n-1}e_1 - h_{n-2}e_2 + h_{n-3}e_3 - \cdots + (-1)^{n-1}e_n.
\end{aligned}$$

Since the sets $\{e_n\}_n$ and $\{h_n\}_n$ are generators for Λ as a k-algebra, they are algebraically independent and so we may define a map $\omega : \Lambda \to \Lambda$ by $\omega(e_n) = h_n$. Then the above relations show that $\omega(h_n) = e_n$, i.e. ω is an involutive automorphism of Λ. Thus the antipode is intimately related to the so-called **fundamental involution** ω.

2.4. **Schur Functions.** We now come to the most important basis for Λ, the so-called Schur functions. We take a combinatorial approach in defining the Schur functions, so let us first recall some combinatorial concepts. Let $\lambda : (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell > 0)$ be a partition, then the **Young diagram** of λ is a diagram of ℓ rows of left-justified boxes, where the i^{th} row has λ_i boxes. We shall use the English convention with Young diagrams and index the rows and columns as we would matrices. The partition λ is the **shape** of the Young diagram.

Let D be a Young diagram of shape λ, then a **semi-standard Young tableau** is an assignment of the boxes of D positive integers such that the rows weakly increase going left to right and the columns strictly increase going top to bottom. A semi-standard Young tableaux is a **standard Young tableau** in which the entries are precisely the integers $1, \ldots, |\lambda|$. The set of all semi-standard Young tableaux on shape λ will be denoted by $\mathsf{SSYT}(\lambda)$ and the set of all standard Young tableaux on shape λ is denoted by $\mathsf{SYT}(\lambda)$. Let T be a semi-standard Young tableaux, then the **content** $c(T)$ of T is the sequence $c(T) = (c_1, c_2, c_3, \ldots)$ in which c_i is the number of boxes of T whose entry is i.

Let μ and λ be Young diagram such that μ is contained in λ. The **skew diagram** λ/μ is the set difference of the diagram of λ with the diagram of μ. A semi-standard skew tableau T of shape λ/μ is a filling of the diagram λ/μ such that the rows weakly increase and the columns strictly increase. The set of all semi-standard skew tableaux of shape λ/μ is denoted by $\mathsf{SSYT}(\lambda/\mu)$.

Let λ be a Young diagram, then the **Schur function** indexed by λ is

$$s_\lambda(\mathbf{x}) \coloneqq \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^{c(T)}.$$

For skew diagram λ/μ, the **skew Schur function** $s_{\lambda/\mu}$ is defined analogously as

$$s_{\lambda/\mu} \coloneqq \sum_{T \in \mathsf{SSYT}(\lambda/\mu)} \mathbf{x}^{c(T)}.$$

For example, let $\lambda = \ydiagram{2,1}$, then

$$\begin{array}{cccccccccccccccc}
s_{\ydiagram{2,1}} = & x_1^2x_2 & + & x_1^2x_3 & + & x_1x_2^2 & + & x_1^2x_4 & + & x_1x_2x_3 & + & x_1x_2x_3 & + & x_2^3 & + & \ldots \\
& \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} & & \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} & & \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} & & \begin{ytableau} 1 & 1 \\ 4 \end{ytableau} & & \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} & & \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} & & \begin{ytableau} 2 & 2 \\ 2 \end{ytableau} & & \ldots.
\end{array}$$

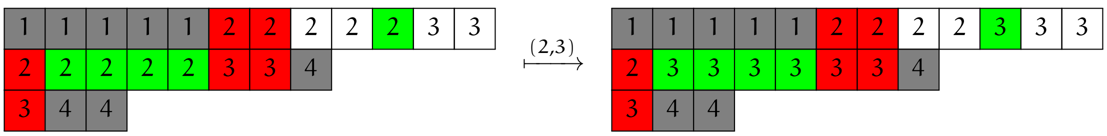
In fact, since the semistandard Young tableaux on λ must be in one of the following forms

$$\begin{ytableau} x & y \\ y \end{ytableau} \qquad \begin{ytableau} x & y \\ z \end{ytableau} \qquad \begin{ytableau} x & z \\ y \end{ytableau}$$

where $x < y < z$, it follows that

$$s_{\yng(2,1)} = m_{\yng(2,1)} + 2m_{\yng(1,1,1)}.$$

2.5. **Schur Functions are Symmetric.** From the definition given so far, it is not clear that the s_λ are actually symmetric functions. To see that the s_λ are indeed symmetric, observe that it shall suffice to show that s_λ is invariant under the action of the transposition $(i, i+1)$, since all such transpositions together will generate $\mathfrak{S}_\infty$. This can be done by finding an involution $\mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\lambda)$. The involution we consider here is due to Bender and Knuth. Let $T \in \mathsf{SSYT}(\lambda)$, then for each column C of T, we may have that both i and $i+1$ appear in C or at most one of i or $i+1$ appear in C. Each pair of i and $i+1$ appearing some the same column will be called a *locked pair* and all other occurrences of i and $i+1$ will be called *free*. The map which we want now is the one which interchanges the number of free i with the number of free $i+1$ in the row. For example, say we are interchanging 2 and 3, then:



The boxes coloured red are the locked boxes. The first row has three free 2s and two free 3s, so after the mapping, the first row has two free 2s and three free 3s; likewise, the second row began with four free 2s and zero free 3s, so after the mapping, the second row has zero free 2s and four free 3s. The content of the first tableau is $(5, 10, 5, 3)$ whereas the content of the second tableau is $(5, 5, 10, 3)$, i.e. the mapping has the effect of interchanging the number of boxes with entry i with that of $i+1$. This mapping is clearly an involution and hence s_λ is invariant under the action of $(i, i+1)$.

2.6. **Hopf Algebra Structure on the Schur Functions.** The wonder of the Schur functions begin to show through when we examine the effect of the Hopf algebra structure on Schur functions. Let us proceed proceed in a strange order and first discuss the action of the antipode on the Schur functions. For a Young diagram λ, denote by λ^t the transpose of the diagram. Then one can show that for skew diagram λ/μ,

$$\omega(s_{\lambda/\mu}) = s_{\lambda^t/\mu^t}, \qquad S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}.$$

One way to see that these identities indeed hold would be through the *Jacobi-Trudi* formulae, which give determinantal formulae for the skew Schur functions with respect to the complete and homogeneous symmetric functions:

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{1 \leqslant i,j \leqslant \ell}, \qquad s_{\lambda^t/\mu^t} = \det(e_{\lambda_i - \mu_j - i + j})_{1 \leqslant i,j \leqslant \ell},$$

where λ has at most ℓ nonzero parts and any function above with a nonpositive subscript is taken to be zero. Some authors define the Schur functions with these formulae and in such definitions, it is immediate that the Schur functions are indeed symmetric.

Next, the coproduct acts wonderfully on the Schur functions: for a Young diagram λ, the coproduct of s_λ is obtained by summing over all subdiagrams of λ:

$$\Delta(s_\lambda) = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu}.$$

One sees this by recalling that Δ introduces two sets of variables, which can be ordered in such a way that $x_1 < x_2 < \cdots < y_1 < y_2 < \ldots$; then the terms occurring in $\Delta(s_\lambda)$ will be all the ways of filling the diagram λ with the x_i and y_j with that ordering.

Finally, we must do some work to understand the multiplicative structure. Since the Schur functions themselves form a basis for Λ, *a priori*, we know that for every triple of partitions μ, λ, ν, there exists coefficients $c^\lambda_{\mu,\nu} \in k$ such that

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

The obvious question now is what are these coefficients and how might we compute them. It turns out that these coefficients $c^{\lambda}_{\mu,\nu}$ are quite important and have a name: they are called the **Littlewood-Richardson coefficients**. The Littlewood-Richardson coefficients turn up in various areas of combinatorics and representation theory, and there are various combinatorial rules to compute them.

Before discussing the Littlewood-Richardson coefficients further, however, let us make some suggestive comments. We have express the coproduct $\Delta(s_\lambda)$ as a linear combination of tensors involving a regular Schur function and a skew Schur function. But skew Schur functions are symmetric and since the regular Schur functions form a basis for Λ, we may express each skew Schur function as a linear combination of regular Schur functions. In this way, for each triple of partitions λ, μ, ν, there are coefficients $\hat{c}^{\lambda}_{\mu,\nu}$ such that

$$\Delta(s_\lambda) = \sum_{\mu,\nu} \hat{c}^{\lambda}_{\mu,\nu} s_\mu \otimes s_\nu.$$

There is no reason to expect that the $c^{\lambda}_{\mu,\nu}$ are related to the $\hat{c}^{\lambda}_{\mu,\nu}$ in any way. Remarkably, it turns out that $c^{\lambda}_{\mu,\nu} = \hat{c}^{\lambda}_{\mu,\nu}$ for all partitions λ, μ, ν. To prove this, we use the *Cauchy identity*:

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}),$$

where the sum on the right is over all partitions λ. There is quite a nice combinatorial proof of this identity. Introduce a new indeterminate t into both sides of the claimed identity, at which one would need to show

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - t x_i y_j} = \sum_{\lambda} t^{|\lambda|} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}).$$

The one expands the left hand side and compare coefficients on both sides with the help of the Robinson-Schensted-Knuth correspondence between semi-standard tableaux and two rowed matrices that are ordered lexicographically.

With Cauchy's identity in hand, we can interpret the coefficients $c^{\lambda}_{\mu,\nu}, \hat{c}^{\lambda}_{\mu,\nu}$ as the coefficients of $s_\mu(\mathbf{x}) s_\nu(\mathbf{y}) s_\lambda(\mathbf{z})$ in the product

$$\begin{aligned}
\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i z_j} \prod_{i,j=1}^{\infty} \frac{1}{1 - y_i z_j} &= \left(\sum_{\mu} s_\mu(\mathbf{x}) s_\mu(\mathbf{y}) \right) \left(\sum_{\nu} s_\nu(\mathbf{y}) s_\nu(\mathbf{z}) \right) \\
&= \sum_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y}) \cdot s_\mu(\mathbf{z}) s_\nu(\mathbf{z}) \\
&= \sum_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y}) \left(\sum_{\lambda} c^{\lambda}_{\mu,\nu} s_\lambda(\mathbf{z}) \right).
\end{aligned}$$

On the other hand, if we now regard the x_i and y_j as lying in the same set of variables,

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i z_j} \prod_{i,j=1}^{\infty} \frac{1}{1 - y_i z_j} = \sum_{\lambda} s_\lambda(\mathbf{x}, \mathbf{y}) s_\lambda(\mathbf{z}) = \sum_{\lambda} \left(\sum_{\mu,\nu} \hat{c}^{\lambda}_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y}) \right) s_\lambda(\mathbf{z}),$$

since the coproduct is exactly the result of considering the Schur function with the two sets of variables $\mathbf{x}$ and $\mathbf{y}$. Rearranging the sums and comparing coefficients yields $c^{\lambda}_{\mu,\nu} = \hat{c}^{\lambda}_{\mu,\nu}$. This property that the structure constants for the product and coproduct being equal is called **self-duality**.

2.7. **Littlewood-Richardson Coefficients.** As we have remarked already, the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu}$ appear in various areas of mathematics. Moreover, there are combinatorial methods, called **Littlewood-Richardson rules**, for computing the coefficients $c^{\lambda}_{\mu,\nu}$. Let us make some brief remarks on these points.

The Littlewood-Richardson coefficients appear in the representation theory of the symmetric group as a multiplicity of a irreducible representation appearing in a restriction representation. Specifically, recall that the irreducible representations of the symmetric group are indexed by partitions μ, ν. Let V_μ and V_ν be the irreducible representations of $S_{|\mu|}$ and $S_{|\nu|}$ corresponding to the partitions μ and ν, respectively. Then the tensor product $V_\mu \otimes V_\nu$ is an irreducible representation of the product group $S_{|\mu|} \times S_{|\nu|}$. For any partition

λ, the Littlewood-Richardson coefficient is the multiplicity of $V_\mu \otimes V_\nu$ in the representation obtained by restricting the irreducible representation V_λ of $S_{|\lambda|}$ to the subgroup $S_{|\mu|} \times S_{|\nu|}$. This phenomenon is a consequence of a deeper connection between Λ and the representations of the symmetric group that is given by the Frobenius characteristic map.

Another representation theoretic appearance of the Littlewood-Richardson coefficients is in the polynomial representations of the general linear group. The polynomial representations of the general linear group are also indexed by partitions μ, ν. It turns out that if V_μ and V_ν are irreducible polynomial representations of the general linear group, then one may decompose the tensor product

$$V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus c^\lambda_{\mu,\nu}}.$$

One reason behind this formula is that the character associated to V_μ is precisely the Schur function s_μ.

The classical Littlewood-Richardson rule asserts that the number $c^\lambda_{\mu,\nu}$ is the number of skew tableaux of shape λ/μ with content ν. For example, for $\lambda = \ydiagram{4,3,2}$, $\mu = \ydiagram{2,1}$ and $\nu = (4, 3, 2)$ we have the following two tableaux:

$$\ytableaushort{\none\none 11, \none 12, 23} \qquad \ytableaushort{\none\none 11, \none 22, 13}$$

which shows that $c^\lambda_{\mu,\nu} = 2$.

2.8. **Further Remarks.** The theory of symmetric functions and this Hopf algebra approach to the symmetric is definitely an object of intrinsic interest. However, it is also nice to know that this theory has many connections to other parts of mathematics. We have touched on a few of the deep connections between Hopf algebras, symmetric functions and representation theory, but there are much deeper connections.

In our discussion of the Schur functions, we saw that Λ is self-dual as a Hopf algebra. Since the Littlewood-Richardson coefficients have a combinatorial meaning, the structure constants $c^\lambda_{\mu,\nu}$ are nonnegative integers. It turns out that graded connected Hopf algebras over $\mathbf{Z}$ with some basis satisfying the self-duality and positivity conditions are rather special. Specifically, Zelevinsky developed a structure theory showing that positive self-dual Hopf algebras are isomorphic to tensor products of scaled versions Λ. In this way, positive self-dual Hopf algebras are, in a sense, polynomials in Λ.

The relationship between combinatorics and Hopf algebras run a lot deeper than we have seen so far. There are a class of Hopf algebras that are very combinatorial in the sense that either: they have a distinguished basis for which the structure constants for the product and coproduct are nonnegative integers; there is a isomorphism between the Hopf algebra and a power series or copower series ring; or the Hopf algebra has some nice representation theoretic properties. Hopf algebras with the precise formulations of these properties are called *combinatorial Hopf algebras*, of which Λ is one of them. It turns out that in the category of combinatorial Hopf algebras, there is a particular object, the *quasi-symmetric functions*, which are like symmetric functions but instead of being indexed by partitions they are indexed by compositions, turn out to be the terminal object there.