Representation theory of finite groups

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Basic representation theory

We wish to discuss the reps of the symmetric groups S_n and the general linear groups $\mathrm{GL}(V)$, among others. Therefore, we need to go over some basic rep theory (mostly of finite groups); I skipped this earlier so we could get right into discussing the reps of $\mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{sl}(3, \mathbf{C})$.

For your convenience and review, I repeat some of the definitions we previously gave for Lie groups and Lie algebras, although they are all identical.

In this section, G denotes a *finite* group. All our rings and algebras will be associative with 1, and their homomorphisms take 1 to 1.

Basic definitions

Definition. A **representation** of G is just a group homomorphism $\rho : G \to \mathrm{GL}(V)$ for V a vector space (here, of course, we need not impose any kind of smoothness condition on ρ as we did for Lie groups).

Remark. As usual, we are only concerned with the case when $\dim V < \infty$, and the field is $\mathbf{C}$. We will also abuse notation and say things like "let V and W be representations of G", or "let $\phi : V \to W$ be an intertwiner of the representations V and W", leaving the homomorphism ρ itself completely implicit. We will also abhor notation such as $\rho(g)(v)$ in favour of the much nicer-looking $g \cdot v$.

Definition. If V is a representation of G, and $W \subseteq V$ is a subspace such that $g \cdot w \in W$ for all $g \in G$ and $w \in W$, we call W an **invariant subspace** (or **subrepresentation**) of V. If V admits no invariant subspaces except $\{0\}$ and V, we say V is an **irreducible representation** (or **irrep**).

Exercise. It turns out that for finite groups (more generally, even for compact Lie groups), *any complex representation V of G decomposes as a direct sum of irreducible representations.* This is because we can start with an arbitrary inner product on V, then "average it over the group" to obtain a new inner product relative to which G acts by unitary operators. With this averaged inner product, orthogonal complements of invariant subspaces can easily be checked to remain invariant, so by iterating this we obtain the aforementioned decomposition.

Definition. If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(W)$ are representations, then by a **morphism of representations from ρ to ρ'** (also called an **intertwiner** or **G-equivariant map**) we mean a linear map $\phi : V \to W$ that commutes with the action of G: to be precise, we have the following eyesore:

$$\phi(\rho(g)(v)) = \rho'(g)(\phi(v)), \qquad \forall g \in G,\ v \in V$$

or, leaving implicit the representations ρ and ρ', we obtain simply

$$\phi(g \cdot v) = g \cdot \phi(v).$$

If we take as objects the representations, and as morphisms the intertwiners, we obtain a category, denoted $\mathbf{Rep}(G)$. We will see later that this category is in fact something familiar. If V and W are representations of G, we write $\mathrm{Hom}_G(V,W) := \mathrm{Hom}_{\mathbf{Rep}(G)}(V,W)$ for the set[1] of all intertwiners $\phi : V \to W$.

Example. From any group action of G on a set X we can obtain a representation in an obvious way. Namely, let V be the free vector space on X, that is let V be spanned by the set $\{e_x : x \in X\}$ of formal symbols, and define a representation of G on V by setting $g \cdot e_x = e_{gx}$ and extending linearly. A representation obtained in this way is called a **permutation representation**.

Since G acts on itself by left multiplication, we obtain from this construction a $|G|$-dimensional representation, called the **left regular representation of** G. This is never irreducible when $G \neq \{1\}$ because the element $\sum_{g\in G} e_g$ clearly spans a one-dimensional invariant subspace.

Schur's Lemma

Given a morphism $\phi : V \to W$ of representations (of groups, or Lie algebras, or quivers...), $\ker\phi$ and $\mathrm{im}\,\phi$ are clearly subreps (of V and W respectively). The following is then immediate.

Lemma (Schur). If ϕ is a morphism between two irreducible representations, then either $\phi = 0$ or ϕ is an isomorphism.

New representations from old

If V and W are representations of G, then all the standard constructions on vector spaces familiar from (multi)linear algebra (e.g. $V^* := \mathrm{Hom}(V,\mathbf{C})$, $V \oplus W$, $V \otimes W$, $\mathrm{Hom}(V,W) = V^* \otimes W$, $\mathrm{Sym}^n V$, $\Lambda^n V$, etc.) yield representations of G in a natural way. For example, $g \in G$ acts on a pure tensor $v \otimes w \in V \otimes W$ by

$$g(v \otimes w) = gv \otimes gw;$$

note carefully that this is *different* from the Lie algebra case, where an element $X \in \mathfrak{g}$ would act on $v \otimes w$ by the "infinitesimal" version of the above, that is, the *product rule*:

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw = (X \otimes I + I \otimes X)(v \otimes w).$$

V^* becomes a rep by the inverse transpose, i.e. given $\rho : G \to \mathrm{GL}(V)$ we define $\rho^* : G \to \mathrm{GL}(V^*)$ by

$$\rho^*(g) := \rho(g^{-1})^T : V^* \to V^*.$$

Again, this is different from the definition for Lie algebras, where we take $\rho^*(X) = -\rho(X)^T$. If you are confused about why these formulas are the "right things" to do, or just want more details, see Fulton and Harris, or any other good book that covers representations of finite groups.

The last construction we feel the need to emphasize is $\mathrm{Hom}(V,W)$. As alluded to above, there is a natural isomorphism $\mathrm{Hom}(V,W) \cong V^* \otimes W$, given merely by sending pure tensors in $V^* \otimes W$ to *rank one linear maps*: that is,

$$V^* \otimes W \ni (\alpha \otimes w) \mapsto (v \mapsto \alpha(v)w) = \langle \alpha, - \rangle w \in \mathrm{Hom}(V,W).$$

Thus, since we already know how to take duals and tensor products of representations, we know how to make $\mathrm{Hom}(V,W)$ a representation as well. Unwrapping this definition, we find that $g \in G$ acts on a linear map $\phi : V \to W$ as follows:

$$(g\phi)(v) = g \cdot \phi(g^{-1} \cdot v).$$

[1] As usual, $\mathrm{Hom}(V,W) = \mathrm{Hom}_{\mathbf{C}}(V,W)$ stands for the set of *all* linear maps $V \to W$, and we often write $\mathrm{End}(V)$ rather than $\mathrm{Hom}(V,V)$; End stands for *endo*morphism.

Modules and algebras

We now introduce *modules*, which are just vector spaces where the scalars are allowed to come from any ring, not necessarily a field. We will see soon how to recast representations as modules.

Note that a vector space over the field k is nothing more than an abelian group M with a ring homomorphism from k to the ring $\mathrm{End}_{\mathbf{Ab}}(M)$ of group homomorphisms $M \to M$. If you have never seen this before, please take a moment and convince yourself that *within this concise statement is encoded the entire list of vector space axioms.* The following definition is therefore nothing surprising.

Let R be a ring.

Definition. A **(left)** R**-module** is an abelian group M together with a ring homomorphism

$$\phi : R \to \mathrm{End}_{\mathbf{Ab}}(M).$$

There is a concept of right R-module, but we will likely not need it.

Remark. We will immediately rid our notation from the morphism ϕ; for $r \in R$ and $m \in M$ we write simply rm for $\phi(r)(m)$.

I will leave it to you to guess the definitions of **homomorphism**, **submodule**, **quotient module**, and so on.

We will find it useful to approach the concept of a module from a slightly more refined angle. For this, let us recall the notion of an algebra over a field k. For us, k will likely always be $\mathbf{C}$.

Definition. An **algebra** over k (or simply k**-algebra**) is a vector space A over k, together with two linear maps

$$\mu : A \otimes A \to A, \qquad \eta : k \to A,$$

called *(vector) multiplication* and *identity* respectively, such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[d, "\cong"] \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}$$

The first diagram encodes the associativity law of the multiplication μ, while the second diagram encodes the fact that the element $\eta(1) \in A$ acts as an identity ("1") for the multiplication μ. Note that the vertical arrows in the second diagram labelled by '$\cong$' are natural isomorphisms (tensoring with the underlying field essentially has no effect, just like taking a Cartesian product of a set with a singleton).

If this definition confuses you at all, remember that linear maps $A \otimes A \to A$ are in one-to-one correspondence with *bilinear* maps $A \times A \to A$. This is the tensor product's *raison d'être*. So μ is just giving us a bilinear way of multiplying the vectors of A, and η is picking out an element of A, namely $\eta(1)$, to be the identity for this multiplication.

If in the definition above, one replaces all vector spaces with sets, all linear maps with functions, and all tensor products $\otimes$ with Cartesian products $\times$, one obtains exactly the definition of a monoid familiar from abstract algebra (i.e. a set with an associative binary operation and an identity element for that operation). Thus, a person who likes category theory might say that *a k-algebra is simply a monoid object in the "monoidal category" of k-vector spaces*, where the monoidal structure is given by the tensor product.

This categorical approach to defining algebras is also nice, because we can then effortlessly flip all the arrows involved and obtain the definition of a "coalgebra". Raymond will speak more about this next week.

We now present the following slight modification of the concept of a module. It consists more of a change of perspective than anything. Let A be an k-algebra, and denote by $\mathrm{End}_k(M)$ the k-algebra of k-linear maps $M \to M$.

Definition. A **(left)** A**-module** (or **representation of** A) is a k-vector space M together with a morphism of k-algebras $\phi : A \to \mathrm{End}_k(M)$.

Group algebras

Having introduced modules, we will now construct from G, as promised above, a ring (more precisely, a $\mathbf{C}$-algebra[2]) whose modules are exactly the same as representations of G.

For inspiration, we present the following cute construction. Given a k-vector space V and a linear operator $T : V \to V$, we can make sense of the operators T, T^2, or even things like $I - T - T^2$, where I is the identity. The assignment $x \mapsto T$ uniquely determines a morphism of k-algebras

$$k[x] \to \mathrm{End}_k(V).$$

Thus V becomes a $k[x]$-module, where the action ("scalar multiplication") is given by

$$(a_0 + a_1 x + \ldots + a_n x^n)v := a_0 v + a_1 T(v) + \ldots + a_n T^n(v)$$

(note that V is a k-vector space, so we already know how to scale by the $a_i \in k$). The $k[x]$-submodules of V are precisely the *T-invariant subspaces W of V* used to develop the theory of canonical forms of matrices, that is, those $W \leq V$ such that $T(W) \subseteq W$ (check this!) In fact, finitely generated modules over principal ideal domains admit a very nice structure theory, and any such V, assuming it is finite-dimensional, certainly satisfies these hypotheses. In fact this structure theory immediately yields the Jordan and rational canonical forms for matrices.

A similar construction turns out to be immensely fruitful in the representation theory of finite groups. Given our finite group G, how can we come up with a $\mathbf{C}$-algebra, call it $\mathbf{C}[G]$, such that the $\mathbf{C}[G]$-modules V (with $\dim_{\mathbf{C}} V < \infty$) correspond exactly to the finite-dimensional reps of G? Well, a rep of G is just a group homomorphism $G \to \mathrm{GL}(V)$. If we had a vector space, call it $\mathbf{C}[G]$, with the elements of G as its basis, then we could just *extend* this to a linear map into $\mathrm{End}_{\mathbf{C}}(V)$, right? At that point, the only question is how to define the algebra structure on $\mathbf{C}[G]$ – but if we want the algebra homomorphisms out of $\mathbf{C}[G]$ to correspond to the group homomorphisms out of G, our hand is forced – the product in $\mathbf{C}[G]$ *must* be given by declaring $e_g e_h = e_{gh}$, where e_g is the basis element corresponding to $g \in G$, and extending bilinearly.

Definition. The **group algebra** (or **group ring**) of G over $\mathbf{C}$, denoted $\mathbf{C}[G]$, is defined to be the free vector space on the set G, that is, $\mathbf{C}[G] = \mathrm{span}_{\mathbf{C}}\{e_g : g \in G\}$.

We can think of $\mathbf{C}[G]$ as the complex vector space of functions $f : G \to \mathbf{C}$. Thus if we define e_g to be the characteristic/indicator function of $\{g\} \subseteq G$, i.e.

$$e_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{else} \end{cases}$$

we may write such an f in the above notation merely as

$$f = \sum_{g \in G} f(g) e_g. \tag{*}$$

Observe, however, that the multiplication on $\mathbf{C}[G]$ as we have defined it above is generally noncommutative. This shows that it is certainly *not* given by the "pointwise" multiplication you might expect on such a space of functions, i.e.

$$(f_1 f_2)(g) = f_1(g) f_2(g).$$

Rather, we see from (*) that elements of $\mathbf{C}[G]$, if we think of them as functions $G \to \mathbf{C}$, are multiplied by *convolution*:

$$(f_1 f_2)(g) = \sum_{a \in G} f_1(a) f_2(a^{-1} g).$$

[2] This goes through, with no modification, for any field k (in fact any ring R), not just $\mathbf{C}$, but we won't bother here.

This is reminiscent of the formula for polynomial multiplication (discrete convolution)

$$(\sum_i a_i x^i)(\sum_j b_j x^j) = \sum_k c_k x^k, \qquad \text{where } c_k = \sum_\ell a_\ell b_{k-\ell}$$

or the convolution of functions familiar from Fourier analysis on the real line:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt.$$

Remark. Note that we can not, from the ring (or algebra) structure of $\mathbf{C}[G]$ alone, recover the group G. This is one motivation for the notion of a Hopf algebra, which are the subject of Raymond's talk next week. For details, see §3.1 of *Representations and Cohomology* (volume 1) by D. J. Benson. Examples of Hopf algebras include group algebras of finite groups, and cohomology algebras of Lie groups.

To summarize our above discussion, the group algebra $\mathbf{C}[G]$ is important to us because given any $\mathbf{C}$-algebra A, there is a one-to-one correspondence between group homomorphisms $G \to A^\times$ and $\mathbf{C}$-algebra homomorphisms $\mathbf{C}[G] \to A$. Here $A^\times$ denotes the group of units (invertible elements) of A.

Somewhat more precisely, one would say, in the language of category theory, that the "group algebra" functor $(G \mapsto \mathbf{C}[G]) : \mathbf{Grp} \to \mathbf{C\text{-}Alg}$ is *left adjoint* to the "group of units" functor $(A \mapsto A^\times) : \mathbf{C\text{-}Alg} \to \mathbf{Grp}$: we have

$$\mathrm{Hom}_{\mathbf{Grp}}(G, A^\times) \cong \mathrm{Hom}_{\mathbf{C\text{-}Alg}}(\mathbf{C}[G], A)$$

naturally in both G and A. For comparison, note that the "free vector space" functor is left adjoint to the forgetful functor from vector spaces to sets.

In particular, take $A = \mathrm{End}_{\mathbf{C}}(V)$. Then $A^\times = \mathrm{GL}(V)$ so the above adjunction reads

$$\mathrm{Hom}_{\mathbf{Grp}}(G, \mathrm{GL}(V)) \cong \mathrm{Hom}_{\mathbf{C\text{-}Alg}}(\mathbf{C}[G], \mathrm{End}_{\mathbf{C}}(V)).$$

This is exactly the correspondence we wanted. We immediately obtain the crucial fact that

a representation of G is the same thing as a module over the complex algebra $\mathbf{C}[G]$.

When we view $\mathbf{C}[G]$ itself as a $\mathbf{C}[G]$-module, the representation of G we obtain is (unsurprisingly) the left regular representation. Also, as one might guess, the direct sum of representations is exactly the direct sum of $\mathbf{C}[G]$-modules. However, the tensor product is a bit more bizarre from this new perspective; its construction relies on the fact that *since we're dealing with a group algebra* we have a natural map $\mathbf{C}[G] \to \mathbf{C}[G] \otimes \mathbf{C}[G]$... you guessed it! Hopf algebras again.

Character theory

We will now associate to a representation a simpler object which surprisingly turns out to capture all of its information.

Definition. The **character** of a representation V of G, denoted χ_V, is defined to be the trace of the action of g on V:

$$\chi_V(g) = \mathrm{tr}(g|_V).$$

Observe immediately that χ_V is constant on each conjugacy class of G, that is $\chi_V(hgh^{-1}) = \chi_V(g)$. We call such functions **class functions**. Also, it is easy to show they play nicely with constructions on representations:

Proposition. We have

$$\chi_{V\oplus W} = \chi_V + \chi_W, \qquad \chi_{V\otimes W} = \chi_V \cdot \chi_W, \qquad \chi_{V^*} = \overline{\chi_V}.$$

Example. If V is a *permutation representation* of G, then $\chi_V(g)$ is clearly the number of fixed points of g as a permutation.

For any finite group G, we can write down a table with one column for each conjugacy class $C \subseteq G$, and one row for each irreducible representation V, where in the (V, C) entry we place the value assumed by χ_V on any element in C. This is called the **character table** of G. We won't elaborate on this here.

We will now show that the characters of irreps form an orthonormal basis for the space of class functions. Suppose V is a representation of a finite group G. Put

$$V^G = \{v \in V : gv = v \text{ for all } g \in G\}.$$

This is the set of **invariants**: vectors in V fixed by the whole action of G. Define a linear map $\phi : V \to V$ by

$$\phi = \frac{1}{|G|} \sum_{g \in G} g.$$

Note that this is the image in $\operatorname{End}_{\mathbf{C}} V$ of the element $|G|^{-1} \sum_{g \in G} e_g \in \mathbf{C}[G]$. When it is convenient, we will frequently abuse notation and ignore the distinction between $e_g \in \mathbf{C}[G]$ and $g \in G$.

Proposition. ϕ is a projection onto V^G.

Proof. Clearly $\operatorname{im} \phi \subseteq V^G$. Conversely, given $v \in V^G$, observe

$$\phi(v) = \frac{1}{|G|} \sum_{g \in G} v = v.$$

Thus $\operatorname{im} \phi = V^G$ and $\phi^2 = \phi$. □

From the above it follows that

$$\dim V^G = \operatorname{tr} \phi = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Here's the trick: given two reps V and W of G, we apply this to the representation $\operatorname{Hom}(V, W) = V^* \otimes W$. From the behaviour of characters on duals and tensors we obtain

$$\chi_{\operatorname{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W.$$

The above then yields

$$\dim \operatorname{Hom}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

Now, one can easily verify that $\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W)$. By Schur's Lemma, if V and W are irreducible, then any intertwiner between them is either 0 or an isomorphism, so

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{else.} \end{cases}$$

More generally, if V is irreducible then $\dim \operatorname{Hom}_G(V, W)$ is the multiplicity of V in W (i.e. the number of copies of V appearing in the decomposition of W into irreps). Similarly if W is irreducible then $\dim \operatorname{Hom}_G(V, W)$ is the multiplicity of W in V. In summary, for irreducible representations V and W, we have that

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{else.} \end{cases}$$

This suggests defining an inner product on the space of complex-valued class functions by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g).$$

This is clearly an inner product. If we do this, the above work shows that:

Proposition. The irreducible characters form an orthonormal set in the space of class functions.

The dimension of this space, of course, is the number of conjugacy classes of G, so the following is immediate.

Corollary. The number of (isomorphism classes of) irreducible representations of G is at most the number of conjugacy classes of G.

In fact, equality holds above (this means the character table is always square).

Corollary. Representations are determined by their characters.

Corollary. A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Corollary. The multiplicity of V_i in V is equal to $\langle \chi_V, \chi_{V_i} \rangle$.

What happens if we apply this to the left regular representation R of G? Remember, it's a permutation representation, so we know its character:

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1. \end{cases}$$

Suppose V is an irreducible representation, and let a be the multiplicity with which it appears in R. Then

$$a = \langle \chi_V, \chi_R \rangle = \frac{1}{|G|} \chi_V(1) \cdot |G| = \dim V.$$

Corollary. Any irrep V of G appears in the left regular representation exactly $\dim V$ times.

Also, writing $R = \bigoplus V_i^{\oplus a_i}$ for distinct irreps V_i, the orthonormality of the irreducible characters also yields

$$\dim R = |G| = \langle \chi_R, \chi_R \rangle = \langle \sum_i a_i \chi_{V_i}, \sum_j a_j \chi_{V_j} \rangle = \sum_{i,j} a_i a_j \langle \chi_{V_i}, \chi_{V_j} \rangle = \sum_i a_i^2 = \sum_i (\dim V_i)^2.$$

Finally, if we evaluate $\chi_R(g)$ for $g \neq 1$ we obtain

$$0 = \sum_i a_i \cdot \chi_{V_i}(g) = \sum_i (\dim V_i) \cdot \chi_{V_i}(g).$$

Given all the irreducible characters except one, the last one is determined by this formula.

Let us now prove that the irreducible characters are indeed an orthonormal *basis* for the space of class functions. It is not difficult to show that:

Proposition. Let $\alpha : G \to \mathbf{C}$ be any function, and for any representation V of G define $\phi_{\alpha,V} : V \to V$ by

$$\phi_{\alpha,V} = \sum_{g \in G} \alpha(g) \cdot g.$$

Then $\phi_{\alpha,V}$ is an intertwiner for all V if and only if α is a class function.

As a consequence, we have

Corollary. The irreducible characters form an orthonormal basis for the space of class functions on G, that is, the number of irreps of G is *equal* to the number of conjugacy classes.

Proof. Suppose $\alpha : G \to \mathbf{C}$ is a class function such that $\langle \alpha, \chi_V \rangle = 0$ for all irreps V. We claim $\alpha = 0$. Consider $\phi_{\alpha,V}$ as defined above; note it's an intertwiner so by Schur's lemma we have $\phi_{\alpha,V} = \lambda I$. Thus if $n = \dim V$ we have

$$\lambda = \frac{1}{n} \operatorname{tr}(\phi_{\alpha,V}) = \frac{1}{n} \sum_{g \in G} \alpha(g) \chi_V(g) = \frac{|G|}{n} \overline{\langle \alpha, \chi_{V^*} \rangle} = 0.$$

Thus $\phi_{\alpha,V} = 0$ or $\sum \alpha(g) \cdot g = 0$ on any rep V of G. Take $V = R$, the regular rep. By the linear independence of $\{g \cdot e_1 : g \in G\} \subseteq R$ we obtain $\alpha(g) = 0$ for all $g \in G$, as required. $\square$

Definition. The **representation ring** of G, denoted $R(G)$, is the free abelian group on the set of isomorphism classes of representations of G, quotiented out by the subgroup generated by all elements of the form $V + W - (V \oplus W)$. It becomes a ring under the tensor product of representations.

Partitions and Young tableaux

Definition. A **partition** of n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\sum \lambda_i = n$. We write $\lambda \vdash n$ to mean "λ is a partition of n", and set $|\lambda| := n$.

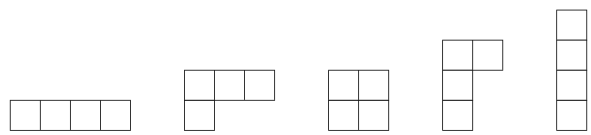
Example. There are 5 partitions of 4:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Here is a common way to represent partitions.

Definition. A **Young diagram** is a collection of boxes arranged in left-justified rows, with a weakly decreasing number of boxes in each row.

Example. The partitions of 4 are represented as



The reason we do this, of course, is so we can put things in the boxes.

Definition. By a **numbering** (or **filling**) of a Young diagram, we mean an assignment of a positive integer to each box. A **(semistandard) Young tableau**, or **SSYT**, is a filling that is weakly increasing in each row, and strictly increasing in each column. If the entries are exactly the numbers from 1 to n, each occurring once, we call it a **standard Young tableau** or **SYT**.

Representations of symmetric groups[3]

The conjugacy classes of S_n are precisely the partitions of n, since two permutations are conjugate if and only if they are of the same cycle structure. We saw that the irreps are in bijection with the conjugacy classes, so it is unsurprising that we can use partitions to classify the irreps of S_n.

Consider the Young tableau T_λ obtained by simply filling the numbers $1, \ldots, n$ in increasing order from left to right, top to bottom. From T_λ we can define two subgroups of S_n:

- The row subgroup P_λ consisting of all $\sigma \in S_n$ that preserve the rows.
- The column subgroup Q_λ consisting of all $\sigma \in S_n$ that preserve the columns.

Clearly $P_\lambda \cap Q_\lambda = \{1\}$. We define the **Young projectors**

$$a_\lambda = \frac{1}{|P_\lambda|} \sum_{\sigma \in P_\lambda} \sigma, \qquad b_\lambda = \frac{1}{|Q_\lambda|} \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) \cdot \sigma.$$

The irreducible representations of S_n are described by the following theorem.

Theorem. The subspace $V_\lambda := \mathbf{C}[S_n]c_\lambda$ of $\mathbf{C}[S_n]$, under left multiplication, is an irrep of S_n. Moreover, every irrep of S_n is isomorphic to V_λ for a unique $\lambda \vdash n$.

The modules V_λ are called **Specht modules**.

[3]Reference: Etingof et al, *Introduction to Representation Theory.*