Representations of $\mathfrak{sl}(3, \mathbf{C})$

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(slides are a work in progress)

Overview

The attack plan

To carry out a similar analysis of the Lie algebra $\mathfrak{g} := \mathfrak{sl}(3, \mathbf{C})$, we will use essentially the same strategy as we did for $\mathfrak{sl}(2, \mathbf{C})$, although some slight generalizations must be made.

Once this is done, however, no further concepts need be introduced to classify all finite-dimensional representations of the remaining semisimple Lie algebras.

Recall that our previous analysis of $\mathfrak{sl}(2, \mathbf{C})$ was based on a decomposition into eigenspaces for the action of the matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A similar thing will work here, although we need to instead look at the action of the subspace $\mathfrak{h} \subset \mathfrak{g}$ of *all* (traceless) diagonal matrices.

Eigenspaces of $\mathfrak{h}$

Let V be a finite-dimensional representation of $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$.

Observation

Commuting diagonalizable matrices are *simultaneously diagonalizable*.

Therefore, as before, V admits a decomposition $V = \bigoplus V_\alpha$, where each $v \in V_\alpha$ is an eigenvector for *every* $H \in \mathfrak{h}$.

Definition

$v \in V$ is called an **eigenvector for** $\mathfrak{h}$ if it is an eigenvector for every $H \in \mathfrak{h}$.

For such a vector v, we can write (note that as usual, the representation is implicit on the LHS):

$$H(v) = \alpha(H) \cdot v \qquad (*)$$

with α depending linearly on $H \in \mathfrak{h}$, that is, $\alpha \in \mathfrak{h}^*$ (here, $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$; it consists of all linear maps $\mathfrak{h} \to \mathbf{C}$).

This motivates us to generalize the concept of eigenvalue as follows.

Definition

$\alpha \in \mathfrak{h}^*$ is called an **eigenvalue for** $\mathfrak{h}$ if there exists a nonzero $v \in V$ such that (*) holds. The **eigenspace associated to** α is defined to be the subspace consisting of all $v \in V$ satisfying (*).

We can recast our previous statement in this new language as follows: any finite-dimensional representation V of $\mathfrak{g}$ decomposes as $V = \bigoplus V_\alpha$ where V_α is an eigenspace for $\mathfrak{h}$ and α runs over a finite subset of $\mathfrak{h}^*$.

Recall the commutation relations we saw for $\mathfrak{sl}(2, \mathbf{C})$,

$$\underbrace{[H, X]}_{\mathrm{ad}(H)(X)} = 2X, \qquad [H, Y] = -2Y.$$

We should interpret these as saying that X and Y are *eigenvectors for the adjoint action* of H on $\mathfrak{sl}(2, \mathbf{C})$. It seems, then, that to continue, we should look for eigenvectors for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}(3, \mathbf{C})$!

Eigenvectors for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$

By taking $V = \mathfrak{sl}(3, \mathbf{C})$ to be the adjoint representation of $\mathfrak{g}$, we can apply our previous remark to obtain a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha} \mathfrak{g}_\alpha \Big) \tag{D1}$$

(here we have just pulled $\mathfrak{h} = \mathfrak{g}_0$ out front) where α runs over a finite subset of $\mathfrak{h}^*$ and we have for any $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}_\alpha$ that

$$[H, Y] = \mathrm{ad}(H)(Y) = \alpha(H) \cdot Y.$$

What kind of matrices $M = (m_{ij})$ could possibly be eigenvectors for $\mathfrak{h}$? Well, if $D = \mathrm{diag}(a_1, a_2, a_3)$ then $(DM)_{ij} = a_i \cdot m_{ij}$ and $(MD)_{ij} = a_j \cdot m_{ij}$, so that

$$[D, M]_{ij} = (DM)_{ij} - (MD)_{ij} = (a_i - a_j)m_{ij}. \tag{**}$$

Observation

$[D, M]$ is a multiple of M for all $D \in \mathfrak{h}$ if and only if $m_{ij} = 0$ for all but one choice of (i, j).

Hence, if we let as usual E_{ij} denote the matrix with 1 in entry (i, j) and 0 everywhere else, the E_{ij} exactly generate the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

Note that since

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

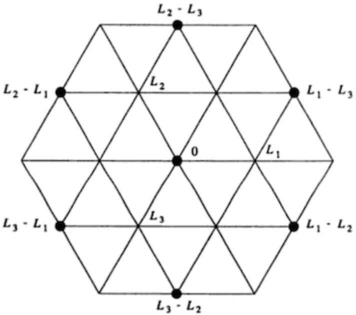
we can realize its dual space $\mathfrak{h}^*$ as the quotient by the annihilator:

$$\mathfrak{h}^* = \mathbf{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3)$$

where L_i sends $\text{diag}(a_1, a_2, a_3) \mapsto a_i$. This is a general fact from linear algebra; if $W \subseteq V$ is a subspace, then W^* is isomorphic to the *quotient* of V^* by the subspace $\text{Ann}(W) \subseteq V^*$ consisting of all linear functionals which vanish on all of W. Indeed, restricting functionals to W yields a (clearly surjective) map $V^* \twoheadrightarrow W^*$ whose kernel is precisely $\text{Ann}(W)$.

The root lattice

In view of calculation (**), there are six functionals $\alpha \in \mathfrak{h}^*$ appearing in the direct sum decomposition of $\mathfrak{g}$, namely $L_i - L_j$ for $i \neq j$. If we draw L_1, L_2 and L_3 in the two-dimensional space $\mathfrak{h}^*$, it is a hexagonal lattice:



We have plotted the six eigenvalues $L_i - L_j \in \mathfrak{h}^*$ on the lattice. The matrix E_{ij} generates the eigenspace $\mathfrak{g}_{L_i - L_j}$. By definition, $\mathrm{ad}(\mathfrak{h})$ maps each $\mathfrak{g}_\alpha$ into itself. Our next task is to determine how the rest of $\mathfrak{g}$ acts.

A concrete way to write down roots

Since we can explicitly exhibit a basis for the space $\mathfrak{h}$, namely

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

it may be more concrete to write the roots $\alpha \in \mathfrak{h}^*$ simply as ordered pairs of complex numbers, in terms of their values on this basis. That is, α may be represented by the ordered pair $(\alpha(H_1), \alpha(H_2))$. If we do this, we see that the roots are:

$$\begin{gathered} (2, -1) \\ (-1, 2) \\ (1, 1) \\ (-2, 1) \\ (1, -2) \\ (-1, -1). \end{gathered}$$

Determining the adjoint action of $\mathfrak{g}_\alpha$ on $\mathfrak{g}$

To do this, let's take some $X \in \mathfrak{g}_\alpha$, some $Y \in \mathfrak{g}_\beta$, and figure out where $\mathrm{ad}(X)(Y)$ lives. Let $H \in \mathfrak{h}$ be arbitrary, and calculate:

$$\begin{aligned} [H, [X, Y]] &= [[H, X], Y] + [X, [H, Y]] && \text{(by Jacobi identity)} \\ &= [\alpha(H) \cdot X, Y] + [X, \beta(H) \cdot Y] \\ &= (\alpha + \beta)(H) \cdot [X, Y]. \end{aligned}$$

Conclusion

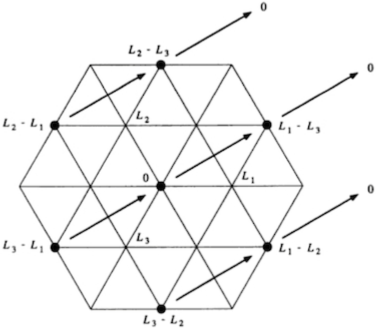
$[X, Y] = \mathrm{ad}(X)(Y)$ is again an eigenvector for $\mathfrak{h}$ with eigenvalue $\alpha + \beta$. Hence,

$$\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}.$$

Notice that this is the same calculation we did for $\mathfrak{sl}(2, \mathbf{C})$ last week. Repeating it almost verbatim, we can show that given *any* representation V, the action of $\mathfrak{g}_\alpha$ carries V_β into $V_{\alpha+\beta}$.

In the adjoint representation, $\mathfrak{g}_\alpha$ therefore acts "by translation" in the sense that it maps the eigenspace corresponding to a dot on the lattice to the eigenspace corresponding to the translation of that dot by α.

For example, here is how $\mathfrak{g}_{L_1-L_3}$ acts:



Similarly to last week (by contriving some subspace W and showing it's invariant), we make the following observation.

Observation

The eigenvalues α occurring in any any irreducible representation V of $\mathfrak{sl}(3, \mathbf{C})$ differ from one another by integral linear combinations of the $L_i - L_j \in \mathfrak{h}^*$.

The lattice in $\mathfrak{h}^*$ generated by the $L_i - L_j$ will be called Λ_R (the **root lattice**).

Definition

The eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on a representation V of $\mathfrak{g}$ is called a **weight** of the representation, the corresponding eigenvectors are called **weight vectors**, and the eigenspaces V_α are called **weight spaces**. The nonzero weights of the adjoint representation are called **roots** of $\mathfrak{g}$, and the $\mathfrak{g}_\alpha \subset \mathfrak{g}$ are called **root spaces**.

Finding the extremal eigenspace

Recall that in our analysis of $\mathfrak{sl}(2, \mathbf{C})$, our next step was to consider a "highest weight vector", that is, a vector in some "extremal" eigenspace V_α.

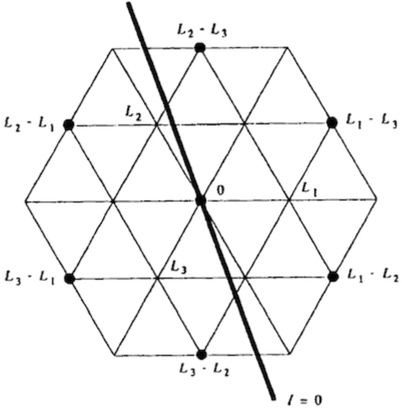
In the current situation this translates to "choosing a direction" and then looking for the furthest α appearing in that direction. To be precise, we select a linear functional $\ell : \Lambda_R \to \mathbf{R}$, extend it by linearity to a linear functional $\ell : \mathfrak{h}^* \to \mathbf{C}$, and then go to the eigenspace V_α for which $\ell(\alpha)$ (or say, its real part) is largest. We'll see the choice of ℓ doesn't matter.

Observation

If we choose ℓ so that its kernel contains a lattice point, it is easy to see that we may run into non-uniqueness of this α. Thus we should choose ℓ to be irrational with respect to the lattice.

Using this strategy we may find a vector $v \in V_\alpha$ that is an eigenvector for $\mathfrak{h}$ and simultaneously killed by all of the "positive" root spaces $\mathfrak{g}_\beta$ (here by "positive" we mean $\ell(\beta) > 0$).

Let's just say we want the kernel of ℓ to look like this (so that the positive root spaces are generated by the E_{ij} with $i < j$):



Then we should choose ℓ to be given by

$$\ell(a_1 L_1 + a_2 L_2 + a_3 L_3) = aa_1 + ba_2 + ca_3$$

with $a > b > c$. For ℓ to be well-defined, we also need $a + b + c = 0$.

Thus for $i < j$, E_{ij} generate the positive root spaces, and E_{ji} generate the negative root spaces. Let

$$H_{ij} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj}.$$

By our work above, we have established:

Lemma

There is a vector $v \in V$ such that

- v is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some α, and
- v is killed by E_{12}, E_{13} and E_{23}.

Such v is called, naturally, a **highest weight vector**.

How do the negative root spaces act on v?

For $\mathfrak{sl}(2, \mathbf{C})$ we saw that, having chosen such a highest weight vector $v \in V_n$, the images of v under successive applications of Y (the generator of the only "negative" root space) then generated V.

Similarly, here we have

Claim

Let V be an irreducible representation of $\mathfrak{sl}(3, \mathbf{C})$ and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive applications of E_{21}, E_{31} and E_{32}.

Proof

Let W be this subspace; we claim W is invariant. Need to show W is preserved by E_{12}, E_{23} and E_{13} (suffices to only check the first two since $E_{13} = [E_{12}, E_{23}]$). Since v is a highest weight vector, it is killed by E_{12}, E_{23} and E_{13}, so that's fine.

Proof (continued)

Now we check that $E_{21}v$ stays in W when E_{12} and E_{23} act on it.

$$E_{12}(E_{21}v) = E_{21}(\underbrace{E_{12}v}_{=0}) + \underbrace{[E_{12}, E_{21}]}_{\in \mathfrak{h}} v = 0 + \alpha([E_{12}, E_{21}])v. \qquad \checkmark$$

$$E_{23}(E_{21}v) = E_{21}(\underbrace{E_{23}v}_{=0}) + \underbrace{[E_{23}, E_{21}]}_{=0} v = 0. \qquad \checkmark$$

Identical computations show that $E_{32}v$ stays in W as well.

Now we use induction. Let w_n be any word of length $\leq n$ in the symbols E_{21} and E_{32} and let W_n be the space spanned by $w_n(v)$ for all such words. Note $W = \bigcup W_n$ since $E_{31} = [E_{32}, E_{21}]$. We claim E_{12} and E_{23} carry W_n into W_{n-1}. Indeed, write w_n as either $E_{21} \circ w_{n-1}$ or $E_{32} \circ w_{n-1}$. Then

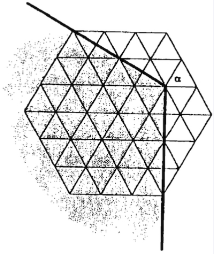
$$\begin{aligned} E_{12}(w_n(v)) = E_{12}(E_{21}(w_{n-1}v)) &= E_{21}(E_{12}w_{n-1}v) + [E_{12}, E_{21}]w_{n-1}v \\ &= E_{21}(W_{n-2}) + \beta([E_{12}, E_{21}])w_{n-1}v \end{aligned}$$

so this lands in W_{n-1}. The other cases are identical; see Fulton & Harris.

Some corollaries of the last claim include:

Corollary

All of the weights $\beta \in \mathfrak{h}^*$ of V lie in a $\frac{1}{3}$-plane with corner α:



Corollary

$\dim V_\alpha = 1$, furthermore the spaces $V_{\alpha+n(L_2-L_1)}$ and $V_{\alpha+n(L_3-L_2)}$ are all at most 1-dimensional; they must be spanned by $(E_{21})^n(v)$ and $(E_{32})^n(v)$ respectively.

In fact, the proof of the claim shows the following.

Proposition

If V is any representation of $\mathfrak{sl}(3, \mathbf{C})$ and $v \in V$ is a highest weight vector, then the subrepresentation W of V generated by the images of v by successive applications of E_{21}, E_{31}, and E_{32} is irreducible.

Proof

Decompose $W = W' \oplus W''$. Projection to W' and W'' commute with the action of $\mathfrak{h}$, so $W_\alpha = W'_\alpha \oplus W''_\alpha$. So one of these spaces is zero, hence v belongs to W' or W'', and hence $W = W'$ or $W = W''$.

As a corollary, each irrep of $\mathfrak{sl}(3, \mathbf{C})$ has a unique highest weight vector up to scaling (thus, a unique one-dimensional subspace of HWVs). [**Note**: this seems to be clear, even without the above – how can two $\frac{1}{3}$-planes with different corners be the same?]

The set of HWVs of a (general) rep will be a union of linear subspaces Ψ_W corresponding to the irreducible subreps W of V, with $\dim \Psi_W$ being the multiplicity of W in the irreducible decomposition of V.

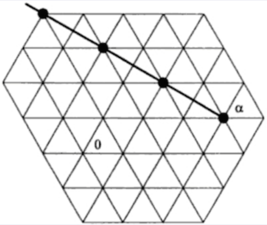
Border vectors

Observation

The vectors $(E_{21})^k(v)$ span a "line" of weight spaces

$$V_\alpha, V_{\alpha+L_2-L_1}, V_{\alpha+2(L_2-L_1)}, \dots$$

Furthermore, this is an uninterrupted string of nonzero eigenspaces $V_{\alpha+k(L_2-L_1)} \cong \mathbf{C}$ until we get to the first m such that $(E_{21})^m(v) = 0$; after that we have $V_{\alpha+k(L_2-L_1)} = 0$ for all $k \geq m$:



How long is this string? The answer to this question (and the rest of this part of our analysis) will simply make use of what we already know about $\mathfrak{sl}(2,\mathbf{C})$.

Observation

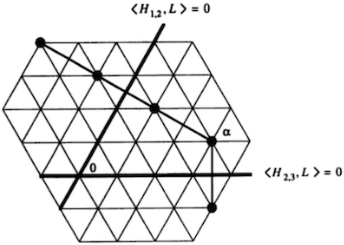
For any $i < j$, the elements E_{ij}, E_{ji} and H_{ij} span a subalgebra $\mathfrak{s}_{L_i - L_j}$ of $\mathfrak{sl}(3,\mathbf{C})$ isomorphic to $\mathfrak{sl}(2,\mathbf{C})$; E_{ij} plays the role of X, E_{ji} that of Y, and their commutator H_{ij} that of H.

In particular, take $(i,j) = (1,2)$. We see that

$$W = \bigoplus_k V_{\alpha + k(L_2 - L_1)}$$

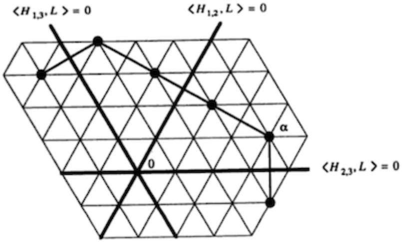
is invariant under the action of $\mathfrak{s}_{L_1 - L_2}$ and thus is a representation of $\mathfrak{s}_{L_1 - L_2} \cong \mathfrak{sl}(2,\mathbf{C})$. Our discussion of $\mathfrak{sl}(2,\mathbf{C})$ then implies that the eigenvalues of H_{12} on W are integral and symmetric about zero!

This means the string of dots must be preserved under reflection in the line $\langle H_{12}, L\rangle = 0$. Looking instead now at the subalgebra $\mathfrak{s}_{L_2 - L_3}$, we see that the string of eigenspaces $V_{\alpha + k(L_3 - L_2)}$ is preserved under reflection in the line $\langle H_{23}, L\rangle = 0$:

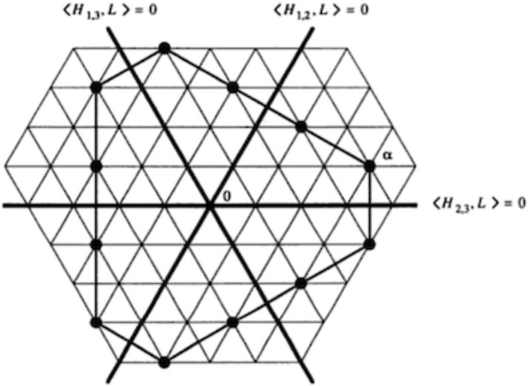


Now consider the last weight space in the first string: let $\beta = \alpha + (m-1)(L_2 - L_1)$ where m is the smallest integer with $(E_{21})^m(v) = 0$. Let $v' \in V_\beta$ be any nonzero vector; observe v' is killed by E_{21}, E_{23} and E_{13}. Thus v' is a highest weight vector if we choose the functional ℓ with $b > a > c$ rather than $a > b > c$!

If we carry out all the analysis with respect to v' instead of v, we make a similar "$\frac{1}{3}$-plane" observation, and also note that the strings of eigenvalues on the lines through β in the $L_1 - L_2$ and $L_3 - L_1$ directions are symmetric about the lines $\langle H_{12}, L\rangle = 0$ and $\langle H_{13}, L\rangle = 0$ respectively.



Anyway, we keep playing this same game until we conclude that the set of weights of V are bounded by a hexagon symmetric with respect to the lines $\langle H_{ij}, L \rangle = 0$ and with one vertex at α.



Proposition

All weights of an irreducible finite-dimensional representation V must be integral linear combinations of the L_i, and be congruent modulo the lattice generated by the $L_i - L_j$.

Now, apply the same strategy as above to other "lines", it follows that all the points in the interior that are congruent to the others modulo the lattice Λ_R must also occur (although we don't know anything about their multiplicities... we will answer that question later):