Lie theory and its ubiquity

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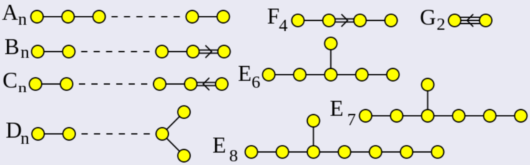
Excerpt from *Introduction to representation theory* (Etingof et al.)



"The graphs listed in the theorem are called (simply laced) **Dynkin diagrams**. These graphs arise in a multitude of classification problems in mathematics, such as classification of simple Lie algebras, singularities, platonic solids, reflection groups, etc. In fact, if we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice!"

Goals of the seminar

At the moment, I am planning to focus mainly on the *structure* and *representation theory* of semisimple Lie algebras. Their full classification, via so-called *Dynkin diagrams*, is one of the most beautiful pieces of modern mathematics.

The purpose of this particular talk is to provide some context (and hence, motivation) for this material by:

- defining the basic objects studied in Lie theory, and
- discussing some of the numerous, oft unexpected, appearances of Lie theory in various diverse branches of mathematics and theoretical physics.

I expect this will very likely convince you to care about it, so please at least bear with me until then (or just skip to the motivation!).

Overview

1. Background and context
2. Motivation

Note

Fully understanding the background/context will require you to know basic differential geometry (smooth manifolds, tangent spaces, pushforward/differential of a smooth map). However, if you are willing to *accept* that Lie algebras are useful without understanding their connection to Lie groups, then I strongly believe such prerequisites are completely unnecessary for everything I do in this seminar. It should mostly be linear algebra.

Lie groups

A Lie group is an object that carries both algebraic structure (it is a group) and geometric structure (it is a smooth manifold), in such a way that the two structures are compatible. This amounts to requiring that the multiplication $(x, y) \mapsto xy$ and inversion $x \mapsto x^{-1}$ are *smooth* maps (a smooth manifold, of course, is precisely the type of geometric object where the concept of a map being "smooth" makes sense). To say it concisely:

Definition

A **Lie group** is a group object in the category of smooth manifolds.

So they are differential-geometric objects. One can also consider group objects in other geometric categories, like the category of algebraic varieties; these are called **algebraic groups** or, a bit less ridiculously, **group varieties**. There are also **complex Lie groups** (replace "smooth" with "complex" above). We will not concern ourselves with these here.

Matrix Lie groups

It turns out that certain families of *matrices* yield many interesting and nontrivial examples of Lie groups. Classically, these were the first studied.

Key Example

Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. We define the $n \times n$ **general linear group over F** by

$$\mathrm{GL}(n, \mathbf{F}) = \{A \in \mathrm{M}_n(\mathbf{F}) : A \text{ invertible}\} = \{A \in \mathrm{M}_n(\mathbf{F}) : \det A \neq 0\}.$$

$\mathrm{GL}(n, \mathbf{F})$ is a group under matrix multiplication, and also (by continuity of the determinant) an open subset of

$$\mathrm{M}_n(\mathbf{F}) \cong \mathbf{F}^{n^2} \cong \begin{cases} \mathbf{R}^{n^2} & \text{if } \mathbf{F} = \mathbf{R} \\ \mathbf{R}^{2n^2} & \text{if } \mathbf{F} = \mathbf{C} \end{cases}$$

and thus it carries the structure of a smooth manifold. The multiplication and inversion (Cramer's rule) are smooth, thus, $\mathrm{GL}(n, \mathbf{F})$ is a Lie group!

Matrix Lie groups

Definition

A **matrix Lie group** is a *closed* subgroup of $\mathrm{GL}(n, \mathbf{F})$.

It is shown in any book on Lie theory that all matrix Lie groups are in fact Lie groups. Here are some particularly important examples.

Examples

- **special linear group**: $\mathrm{SL}(n, \mathbf{F}) = \{A \in \mathrm{GL}(n, \mathbf{F}) : \det A = 1\}$ (linear operators which preserve volume and orientation).
- **orthogonal group**: $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbf{R}) : A^T A = I\}$ (linear *isometries* of $\mathbf{R}^n$, i.e. preserve the standard distance).
- **unitary group**: $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbf{C}) : A^* A = I\}$ (linear isometries of $\mathbf{C}^n$).

Matrix Lie groups

Many of these examples come from looking at all the invertible linear operators on a space V that preserve some sort of "extra structure" on V, for example, some kind of bilinear, hermitian, or symplectic form. It could also be a quadratic form, or something fancier.

More Examples

- **special orthogonal group**: $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbf{R})$ (orientation-preserving linear isometries of $\mathbf{R}^n$).
- **special unitary group**: $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbf{C})$ (orientation-preserving linear isometries of $\mathbf{C}^n$).
- **symplectic group** $\mathrm{Sp}(n)$.
- the group of upper triangular matrices.
- the **Heisenberg group** $\mathrm{H}_3(\mathbf{R})$, important in quantum mechanics.

Are all Lie groups matrix Lie groups?

Since all of the classical groups are matrix Lie groups, one might ask whether every Lie group is a matrix Lie group (up to isomorphism).

Interestingly, this is **not** the case: the universal covering group of $\mathrm{SL}(2, \mathbf{R})$ is one example. Another is given by the so-called *metaplectic groups* Mp_{2n}, which double cover the symplectic groups.

Remark

As with most mathematical objects, there are standard operations which produce new Lie groups from given ones. Sometimes, such a Lie group may actually be isomorphic to a matrix Lie group, even though this is far from obvious from its abstract specification. An example is the groups $\mathrm{Spin}(n)$, which double cover the groups $\mathrm{SO}(n)$; these can be constructed as certain elements in a Clifford algebra. We will likely explore this more later.

Lie algebras

We now consider linear-algebraic objects called Lie algebras. For us, these will take the center stage. First, we will define them abstractly, and then discuss how to attach one to any Lie group.

Definition

A **Lie algebra** $\mathfrak{g}$ is a vector space equipped with a **Lie bracket**, that is, a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

- **alternating**: $[X, X] = 0$ for all $X \in \mathfrak{g}$.
- **Jacobi identity**: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

From the alternating property we see that $[X, Y] = -[Y, X]$, so $[-,-]$ is **skew-symmetric**. In general, $[-,-]$ is not associative; the Jacobi identity is a kind of "substitute" for associativity.

The Lie algebra of a Lie group

Note

For those with no manifold background, this may not be comprehensible. Don't worry if you can only understand it intuitively; there is a considerably simpler definition for matrix Lie groups.

Let G be a Lie group. Denote by $\Gamma(TG)$ the space of all vector fields on G. Then $\Gamma(TG)$, under the usual Lie derivative $[-,-]$ of vector fields, forms an infinite-dimensional real Lie algebra. Let $\mathfrak{g}$ denote the subspace of all *left-invariant*[1] vector fields $X \in \Gamma(TG)$. One can verify that $\mathfrak{g}$ is a Lie subalgebra of $\Gamma(TG)$.

Next, let T_eG be the tangent space to the identity of G; recall $\dim T_eG$ is equal to the manifold dimension of G. We define a linear $\ell : T_eG \to \mathfrak{g}$ just by using the pushforward of the left-multiplication maps $L_g : G \to G$ to "create a copy" of a tangent vector $X \in T_eG$ in each space T_gG.

[1] These are vector fields which loosely speaking "look the same" everywhere.

The Lie algebra of a Lie group

This map $\ell : T_e G \to \mathfrak{g}$ is a linear isomorphism; its inverse is the linear map $\mathfrak{g} \to T_e G$ given by $X \mapsto X_e$ (that is, start with a left-invariant vector field X on G, and look at what it does at the identity element).

Thus, we can use ℓ to "transport" the Lie algebra structure of $\mathfrak{g}$ to $T_e G$. Explicitly, we define $[X, Y] := [\ell(X), \ell(Y)]_e$ for $X, Y \in T_e G$.

This turns $T_e G$ into a *finite-dimensional* Lie algebra;

$$\dim \mathfrak{g} = \dim T_e G = \dim G.$$

We henceforth identify $\mathfrak{g} \cong T_e G$. The Lie algebra $\mathfrak{g}$, being a linear object, is easier to understand and is therefore very useful in studying G. The structure of G near the identity element is very much controlled by $\mathfrak{g}$. The stronger the connectedness assumptions on G, the more global and pronounced this "control" becomes. We elaborate on this below.

The matrix exponential

Although the above definition of the Lie algebra $\mathfrak{g}$ is the most general since it applies to *all* Lie groups G, the situation simplifies considerably for matrix Lie groups. First, we recall the matrix exponential.

Definition

Let $X \in \mathrm{M}_n(\mathbf{F})$. We define the **exponential of the matrix** X by

$$\exp X = e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots .$$

The following calculation shows absolute convergence of this series, so life is good:

$$\sum_{k=0}^{\infty} \left\| \frac{X^k}{k!} \right\| = \lim_{N \to \infty} \sum_{k=0}^{N} \left\| \frac{X^k}{k!} \right\| \leq \lim_{N} \sum_{k=0}^{N} \frac{\|X\|^k}{k!} = \sum_{k=0}^{\infty} \frac{\|X\|^k}{k!} = \exp \|X\|.$$

The Lie algebra of a *matrix* Lie group

The following is proved in any introductory book on Lie groups.

Theorem

Let $G \leq \mathrm{GL}(n, \mathbf{F})$ be a matrix Lie group, and $\mathfrak{g}$ its Lie algebra. Then

$$\mathfrak{g} \cong \{X \in \mathrm{M}_n(\mathbf{F}) : \exp(tX) \in G \text{ for all } t \in \mathbf{R}\}.$$

Here, the Lie bracket on $\mathrm{M}_n(\mathbf{F})$, and hence on the RHS above, is just the commutator of matrices: $[X, Y] = XY - YX$.

A Lie group homomorphism $\varphi : (\mathbf{R}, +) \to G$ (by this, we mean a smooth group homomorphism) is referred to as a **one-parameter subgroup** of G. To each such φ there corresponds a unique $X \in \mathfrak{g}$, called the **infinitesimal generator** of φ, so that $\varphi(t) = \exp(tX)$, $\forall t \in \mathbf{R}$.

So the above may be interpreted in the following slightly more geometric sense: *the Lie algebra $\mathfrak{g}$ of a matrix Lie group G consists precisely of all infinitesimal generators of one-parameter subgroups of G.*

To what extent does the Lie algebra control the Lie group?

The following result is of significant theoretical and practical importance.

Theorem (Baker-Campbell-Hausdorff)

Let G be a Lie group. For $X, Y \in \mathfrak{g}$, we have

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots$$

It is not supposed to be obvious what "…" refers to. The point is that the right-hand side involves only X and Y, brackets of X and Y, brackets of these, and so on. This has the following profound corollary.

Corollary, at least in the case of matrix Lie groups

Let G and H be Lie groups, with G *simply connected*. There is a (nice) one-to-one correspondence between Lie group homomorphisms $\Phi : G \to H$ and Lie algebra homomorphisms $\phi : \mathfrak{g} \to \mathfrak{h}$.

Representation theory

Representations are to vector spaces as group actions are to sets. That is, a representation of a group (or Lie group, or Lie algebra) is just a *linear* action on a vector space.

Definition

Let G be a Lie group. A **finite-dimensional representation** of G is a Lie group homomorphism $\Pi : G \to \mathrm{GL}(V)$ for some vector space V, with $0 < \dim V < \infty$.

Since $\mathrm{M}_n(\mathbf{F})$ is the Lie algebra of $\mathrm{GL}(n, \mathbf{F})$, we write $\mathfrak{gl}(n, \mathbf{F}) := \mathrm{M}_n(\mathbf{F})$.

Definition

Let $\mathfrak{g}$ be an (abstract) Lie algebra. A **finite-dimensional representation** of $\mathfrak{g}$ is a morphism of Lie algebras[2] $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V, with $0 < \dim V < \infty$.

[2] A morphism of Lie algebras is just a linear map preserving the Lie bracket: $\pi([X, Y]) = [\pi(X), \pi(Y)]$.

Representation theory in general

Representation theory is not limited to Lie groups and Lie algebras. It also studies finite groups, quivers, partially ordered sets, and perhaps most fundamentally, *associative algebras*.

Thanks to some convenient adjunctions, such as

$$\mathrm{Hom}_{\mathbf{Grp}}(G, A^{\times}) \cong \mathrm{Hom}_{K\text{-}\mathbf{Alg}}(K[G], A),$$

it turns out that *all of these seemingly different flavours of representation theory can be recast merely as the representation theory of some cleverly conceived associative algebra.*

Examples include the *group algebra* of a finite group, the *path algebra* of a quiver, and the *universal enveloping algebra* of a Lie algebra. Since a representation of an associative algebra A is the same thing as an A-module, this enables us to employ the techniques of module theory.

Representation theory

With some care, we can also define infinite-dimensional representations. Here is an example from harmonic analysis where these show up.

Example

$\mathbf{R}$, SO(2) and SO(3) are Lie groups. They admit (infinite-dimensional) representations on the Hilbert spaces of "square-integrable" functions $L^2(\mathbf{R})$, $L^2(S^1)$, and $L^2(S^2)$, respectively, by acting as translations: more explicitly, for $g \in G$ and $\varphi \in L^2$, we define

$$(g * \varphi)(x) = \varphi(g^{-1} \cdot x).$$

For example, $a \in \mathbf{R}$ acts on $L^2(\mathbf{R})$ by $(a * \varphi)(x) = \varphi(x - a)$.

Decomposing these representations yields the theories of Fourier transform, Fourier series, and spherical harmonics, respectively.

Lie theory in physics

Example

In the standard model of particle physics, the *Eight-Fold Way* involves an 8-dimensional representation of the Lie algebra $\mathfrak{sl}(3; \mathbf{C})$. This will be one of the first Lie algebras whose representation theory we study.

Example

The representation theory of a particular (non-compact) Lie group known as the *Poincaré group* plays a crucial role in quantum field theory.

There are many, many more instances where the representation theory of these objects arises in particle physics.

Example

Mysterious objects known as *quantum groups* (particular special cases of *Hopf algebras*) arise as "deformations" of universal enveloping algebras.

Lie theory in physics

There are also intimate connections to differential equations, and special functions/orthogonal polynomials. We will probably see at least some of this.

Excerpts from *Lie Groups, Physics, and Geometry* (Gilmore)

"If finite groups were required to decide on the solvability of finite-degree polynomial equations, then "infinite groups" would probably be involved in the treatment of ordinary and partial differential equations."

"... most of the classical functions of mathematical physics are matrix elements of simple Lie groups, in particular matrix representations. There is a very rich connection between Lie groups and special functions that is still evolving."

The five so-called "exceptional simple Lie groups", which do not fit into the classification, are largely related to octonionic symmetry. We may see a bit of these as well.

Lie theory in geometry and topology

Example

One can discuss *covering groups* of Lie groups; these have interesting relationships to *projective representations*.

Example

Lie algebras are also an important ingredient in the Chern-Weil theory of characteristic classes of vector bundles.

Example

There are applications of the representation theory of Lie algebras to projective algebraic geometry. A good reference for this is Fulton and Harris.

Lie theory in combinatorics

There is considerable interplay between the representation theory of Lie algebras and combinatorics.

Example

The Schur functions, which are examples of symmetric functions (which in turn are of significant interest in enumeration), arise in the representation theory of the unitary groups.

Example (Problem 6.25 of R. Stanley's *Enum. Comb. Volume 2*)

The Catalan numbers have many algebraic interpretations. In particular, they arise as the dimensions of irreducible representations of the symplectic group.

Other examples include quantum walks on graphs, crystal operators, palindromic unimodal sequences, q-theory... the list goes on and on.

References

Here are some relevant books:

- Bump, *Lie Groups*.
- Fulton and Harris, *Representation Theory: A First Course*.
- Gilmore[3], *Lie Groups, Physics, and Geometry*.
- Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*.
- Humphreys, *Introduction to Lie Algebras and Representation Theory*.
- Varadarajan, *Lie Groups, Lie Algebras, and their Representations*.

[3] Be warned; this guy claims at some point that if H is a normal subgroup in G, then $G \cong H \times (G/H)$.