

Derived Functors

Idea: Suppose we have $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ & suppose that F is a right-exact additive functor. (Eg. In $R\text{-Mod}$, if M is a right- R -module then $F = M \otimes -$ is a right exact functor from $R\text{-Mod}$ to Ab .) We know

$$0 \rightarrow K \rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \rightarrow 0$$

for some K . We'd like to understand K . (Eg. If $N_1 \xrightarrow{i} N_2$ in $R\text{-Mod}$, what is $\ker(M \otimes_R N_1 \xrightarrow{id \otimes i} M \otimes_R N_2)$ As we'll see, there is a first left-derived functor $L_1 F$ such that

$$L_1 F C \xrightarrow{\delta} FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \rightarrow 0.$$

The object is independent of maps — just needs $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Def] Suppose that \mathcal{C}, \mathcal{D} are abelian categories, $F: \mathcal{C} \rightarrow \mathcal{D}$ is additive and right-exact. We also need that \mathcal{C} has enough projectives. Then if $A \in \text{Ob}(\mathcal{C})$,

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

there exists a projective resolution of A

$$P_i: \dots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0 \rightarrow 0 \rightarrow \dots$$

From this we got a chain complex P . Apply F :

$$\dots \rightarrow FP_2 \xrightarrow{Fd_2} FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{Fd_0} 0 \rightarrow \dots$$

Then we define

$$L_i F A := H_i(FP).$$

L_i is called the i^{th} left-derived functor of F .

Why is this well-defined? If P_0 and P'_0 are two chain complexes coming from projective resolutions of A , then $\exists u: P_0 \rightarrow P'_0, v: P'_0 \rightarrow P_0$ with $v \circ u \sim id_{P_0}, u \circ v \sim id_{P'_0}$.

$$\begin{array}{ccccccc} \dots & \rightarrow & P_i & \rightarrow & P_{i-1} & \rightarrow & A \rightarrow 0 \\ & & \downarrow u & & \downarrow u & & \downarrow id \\ \dots & \rightarrow & P'_i & \rightarrow & P'_{i-1} & \rightarrow & A \rightarrow 0 \end{array}$$

$$F(u): FP_i \rightarrow FP'_i, \quad F(v): FP'_i \rightarrow FP_i$$

$$F(u) \circ F(v) = F(u \circ v) \sim F(id_{P'_i}) = id_{FP'_i}$$

Similarly for $F(v) \circ F(u)$. So $F(u)$ gives a quasi-isomorphism, i.e.

$H_i(FP_i) \cong H_i(FP'_i)$. So $L_i F(A)$ is well-defined.

If $f: A \rightarrow B$, then what is $L_i Ff$?

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & A \rightarrow 0 \\ & & \downarrow d_1 & & \downarrow d_0 & & \downarrow f \\ \cdots & \rightarrow & Q_1 & \rightarrow & Q_0 & \rightarrow & B \rightarrow 0 \end{array}$$

$\exists \theta: P_i \rightarrow Q_i$ st θ induces f on $H_0(P_i) \rightarrow H_0(Q_i)$

Then $L_i F(f)$ is just the map from $H_i(FP_i) \rightarrow H_i(FQ_i)$ induced by $F(\theta): FP_i \rightarrow FQ_i$. Check this is well-defined.

We saw $L_i FA$ is independent of choice of projective resolution. We also have

Theorem 1: $L_0 F = F$

Proof: $\exists \varphi: P_0 \rightarrow A$.

$$\cdots P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi} A \rightarrow 0$$

F is right exact so if $K = \ker(P_0 \xrightarrow{\varphi} A)$ $0 \rightarrow K \rightarrow P_0 \rightarrow A \rightarrow 0$
 $\Rightarrow FK \rightarrow FP_0 \rightarrow FA \rightarrow 0$ exact.

In fact $P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ is exact $FP_1 \rightarrow FP_0 \rightarrow FA \rightarrow 0$ exact

What is $L_0 F(A)$? $P_1 \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$ Apply F

$$FP_1 \rightarrow FP_0 \rightarrow 0 \rightarrow \cdots$$

& $L_0 F = 0^{\text{th}}$ homology $FP_0 / \text{Im}(FP_1)$

But $\text{Im}(FP_1) = \ker(FP_0 \rightarrow FA)$

Mitchell's embedding allows us to work in $R\text{-Mod}$

$$FP_0 \xrightarrow{\varphi} A \rightarrow 0$$

$$FP_0 / \ker \varphi \cong A$$

$$\text{" } L_0 FA$$

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Theorem 2: If A is projective then $L_i FA = 0 \forall i \geq 1$

Proof: Notice $0 \rightarrow 0 \rightarrow A \xrightarrow{id} A \rightarrow 0$ is a projective resolution of A

Apply F : $\cdots \rightarrow F0 \rightarrow F0 \rightarrow \cdots \rightarrow F0 \rightarrow FA \rightarrow F0 \rightarrow \cdots$

$\because F0 = 0$ We get

$$0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow FA \rightarrow 0 \rightarrow \cdots$$

So $H_i(FP_i) = 0 \forall i \geq 1$ where $P_i \cdots \rightarrow 0 \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$

So $L_i FA = 0 \forall i > 0$

Theorem 3: If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short-exact sequence then \exists a long exact sequence

$$\begin{array}{ccccccc}
 & & L_0FA & & L_0FB & & L_0FC \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \delta_1 \rightarrow & FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC & \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \delta_2 \rightarrow & L_1FA & \xrightarrow{L_1Ff} & L_1FB & \xrightarrow{L_1Fg} & L_1FC & \rightarrow \dots
 \end{array}$$

with $\delta_i: L_iFC \rightarrow L_{i-1}FA$.

Idea:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \dots \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & A \rightarrow 0 \\
 & \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow \theta_0 & & \downarrow f \\
 \dots \rightarrow & U_2 & \rightarrow & U_1 & \rightarrow & U_0 & \rightarrow & B \rightarrow 0 \\
 & \downarrow \tau_2 & & \downarrow \tau_1 & & \downarrow \tau_0 & & \downarrow g \\
 \dots \rightarrow & Q_2 & \rightarrow & Q_1 & \rightarrow & Q_0 & \rightarrow & C \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & & 0
 \end{array}$$

Like to find $U_0 \rightarrow B \rightarrow 0$ st $0 \rightarrow P_0 \xrightarrow{\theta} U_0 \xrightarrow{\tau} Q_0 \rightarrow 0$ exact.

Then claim $0 \rightarrow FP_i \rightarrow FU_i \rightarrow FQ_i \rightarrow 0$ is exact $\forall i$.

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Remark: $F: C \rightarrow D$ abelian categories, F additive $\Rightarrow F(0_c) = 0_d$.

$$\begin{array}{ccc}
 0_c & \xrightarrow{id} & 0_c \\
 \downarrow & & \downarrow \\
 F(0_c) & \xrightarrow{id} & F(0_c)
 \end{array}$$

Show $F(0_c)$ is initial and terminal

Proof (of thm 3): We reduced this to two claims:

1) Given projective resolutions

$$\begin{array}{ccc}
 P_\bullet & \rightarrow & A \rightarrow 0 \\
 Q_\bullet & \rightarrow & C \rightarrow 0
 \end{array}$$

\exists a projective resolution $U_\bullet \rightarrow B \rightarrow 0$ & chain maps $\theta: P_\bullet \rightarrow U_\bullet$
 $\tau: U_\bullet \rightarrow Q_\bullet$

st $0 \rightarrow P_i \xrightarrow{\theta} U_i \xrightarrow{\tau} Q_i \rightarrow 0$ is exact & θ induces $f: A \rightarrow B$
on $H_0(P_\bullet) \xrightarrow{f} H_0(U_\bullet)$ & τ induces $g: B \rightarrow C$ on $H_0(U_\bullet) \rightarrow H_0(Q_\bullet)$

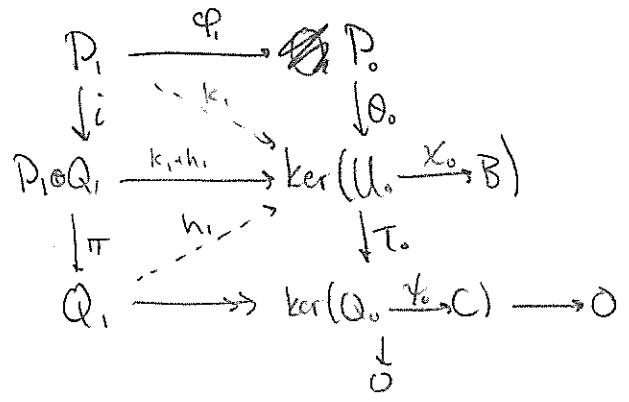
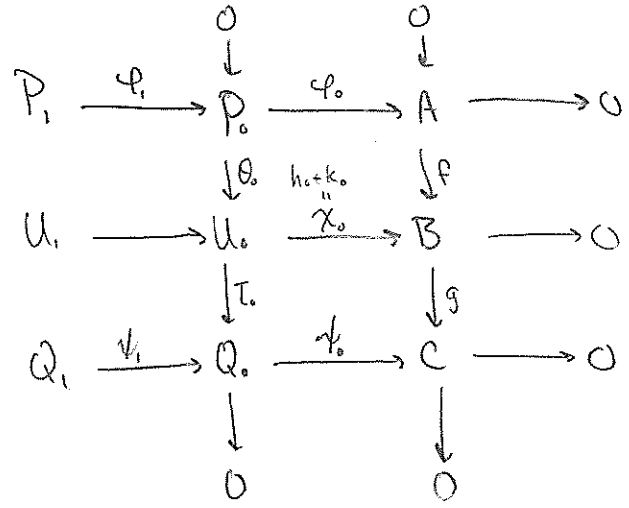
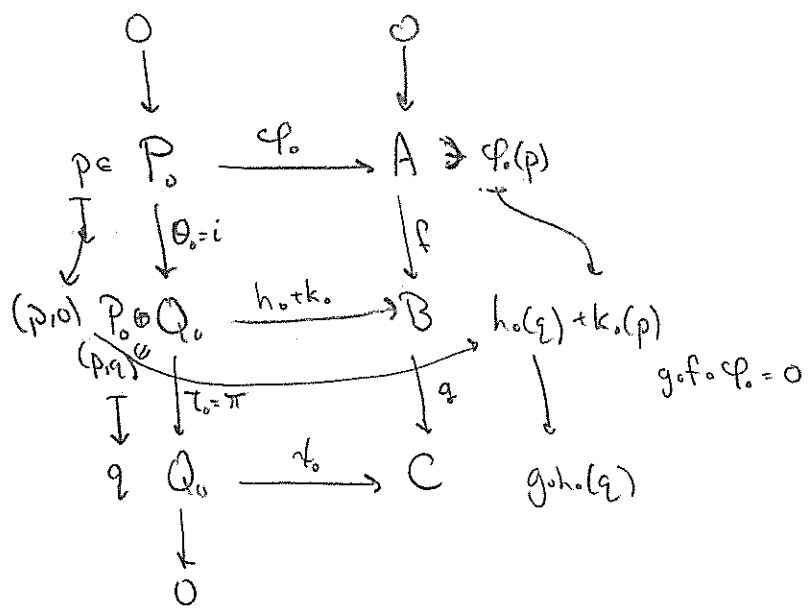
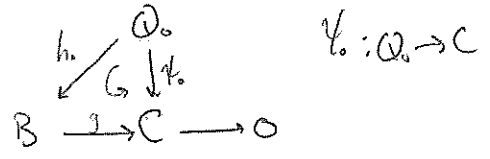
$0 \rightarrow P_0 \rightarrow U_0 \rightarrow Q_0 \rightarrow 0$ exact, P_0, U_0, Q_0 proj.

$\Rightarrow 0 \rightarrow FP_0 \rightarrow FU_0 \rightarrow FQ_0 \rightarrow 0$

Verification of claim 1:

$\exists h_0 : Q_0 \rightarrow B$ st $g \circ h_0 = \psi_0$

Let $k_0 : P_0 \rightarrow B, k_0 = f \circ \varphi_0$



continuing in this way gives the result. (ie gives claim 1)

Verification of Claim 2:

It is sufficient to show that if $0 \rightarrow P \rightarrow U \rightarrow Q \rightarrow 0$ is a short exact sequence of proj. ob then

$$0 \rightarrow FP \rightarrow FU \rightarrow FQ \rightarrow 0 \text{ is exact}$$

Why?

$$\begin{array}{ccccccc}
 & & \xrightarrow{t} & & \xrightarrow{s} & & \\
 & & \searrow & & \swarrow & & \\
 0 & \rightarrow & P & \xrightarrow{\theta} & U & \xrightarrow{\tau} & Q \rightarrow 0 \\
 & & & & & & \downarrow \text{Id} \\
 & & & & & & Q \text{ proj}
 \end{array}$$

$$\begin{array}{ccc}
 & U & \\
 t \swarrow & & \downarrow \text{Id} - s \cdot \tau \\
 P & \xrightarrow{\theta} & \text{Im } \theta \\
 & & \text{ker}(\tau)
 \end{array}$$

Apply F , F is right-exact.

There was a lot of confusion but apparently it's easy.

Theorem 4: $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$

$$\begin{array}{ccccccc}
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \rightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \rightarrow 0
 \end{array}$$

exact rows & commutes

$$\begin{array}{ccc}
 L_{i+1}FC & \xrightarrow{\delta_{i+1}} & L_iFA \\
 \downarrow L_{i+1}F\alpha & & \downarrow L_iF\beta \\
 L_{i+1}FC' & \xrightarrow{\delta'_{i+1}} & L_iFA'
 \end{array}$$

commutes

Remark: Similarly, if $G: C \rightarrow D$ is additive & left-exact
 If $C \in \text{Ob}(\mathcal{C})$, If G has enough inj.

$$0 \rightarrow C \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

Apply G $0 \rightarrow GC \rightarrow GI^0 \rightarrow GI^1 \rightarrow \dots$

$$0 \rightarrow GI^0 \rightarrow GI^1 \rightarrow \dots \text{ chain complex}$$

$$H^i(GI^\bullet) =: R^iG(C)$$

We have 1) $R^0G = G$

2) C inj $\Rightarrow R^iG(C) = 0 \quad \forall i > 0$

3) $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ exact

$$0 \rightarrow GA \rightarrow GB \rightarrow GC \xrightarrow{\delta^0} R^1GA \rightarrow R^1GB \rightarrow R^1GC \rightarrow \dots$$

$$4) 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

$$0 \rightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \rightarrow 0$$

$$R^i G C \xrightarrow{(S^i)'} R^{i+1} G A$$

$$\downarrow R^i G \gamma \quad \downarrow R^{i+1} G \alpha$$

$$R^i G C' \xrightarrow{(S^i)'} R^{i+1} G A'$$

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Remark: Notice thms 1-4 allow us to recover $L_i F A \forall i \geq 0$ \forall objects A
 $L_0 F A = F A \checkmark$

If we know $L_i F A \forall i < n$

$$0 \rightarrow \overset{\text{kernel}}{K} \xrightarrow{\text{projective}} P \rightarrow A \rightarrow 0$$

$$\hookrightarrow F K \rightarrow F P \rightarrow F A \rightarrow 0$$

$$\hookrightarrow L_i F K \rightarrow 0 \rightarrow L_i F A \rightarrow$$

$$\hookrightarrow L_{i+1} F K \rightarrow 0 \rightarrow L_{i+1} F A \rightarrow$$

$$\therefore L_i F P = 0 \forall i > 0$$

$$L_2 F A \cong L_1 F K, L_3 F A \cong L_2 F K$$

$$* 0 \rightarrow L_i F A \rightarrow F K \rightarrow F P \rightarrow F A \rightarrow 0$$

So knowing $L_i F K$ gives us $L_{i+1} F A \forall i \geq 1$ & we can get $L_1 F A$ from (*)

recursive procedure for computing functors