Derived Functors

Idea: Suppose we have $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ & suppose that F is a right-exact additive functor. (Eg. In R-Mod, if M is a right-R-module then $F = M \otimes -$ is a right exact functor from R-Mod to Ab.) We know

$$0 \to K \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$$

for some K. We'd like to understand K. (Eg. If $N_1 \xhookrightarrow{i} N_2$ in R-Mod, what is $\ker(M \otimes_R N_1 \xrightarrow{id \otimes i} M \otimes_R N_2)$) As we'll see, there is a first left-derived functor L_1F such that

$$L_1FC \xrightarrow{\delta} FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0.$$

The object is independent of maps — just needs $0 \to A \to B \to C \to 0$.

<u>Def</u> Suppose that $\mathcal{C}, \mathcal{D}$ are abelian categories, $F : \mathcal{C} \to \mathcal{D}$ is additive and right-exact. We also need that $\mathcal{C}$ has enough projectives. Then if $A \in Ob(\mathcal{C})$,

$$\cdots P_2 \to P_1 \to P_0 \to A \to 0$$

there exists a projective resolution of A

$$P_\cdot : \cdots P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} 0 \to 0 \to \cdots$$

From this we got a chain complex $P_\cdot$. Apply F:

$$\cdots \to FP_2 \xrightarrow{F\varphi_2} FP_1 \xrightarrow{F\varphi_1} FP_0 \xrightarrow{F\varphi_0} 0 \to \cdots$$

Then we define

$$L_iFA := H_i(FP_\cdot).$$

L_i is called the i^{th} left-derived functor of F.
Why is this well-defined? If $P_\cdot$ and $P'_\cdot$ are two chain complexes coming from projective resolutions of A, then $\exists\, u : P_\cdot \to P'_\cdot$, $v : P'_\cdot \to P_\cdot$ with $v \circ u \sim id_{P_\cdot}$, $u \circ v \sim id_{P'_\cdot}$.

$$\begin{array}{ccccccc} \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow 0 \\ & \downarrow u_1 & & \downarrow u_0 & & \downarrow id & \\ \cdots \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A & \longrightarrow 0 \end{array}$$

$$F(u) : FP_\cdot \to FP'_\cdot, \quad F(v) : FP'_\cdot \to FP_\cdot$$

$$F(u) \circ F(v) = F(u \circ v) \sim F(id_{P'_\cdot}) = id_{FP'_\cdot}$$

Similarly for $F(v) \circ F(u)$. So $F(u)$ gives a quasi-isomorphism, ie. $H_i(FP_\cdot) \cong H_i(FP'_\cdot)$. So $L_iF(A)$ is well-defined.
If $f : A \to B$, then what is L_iFf?

$$\begin{array}{ccccccc} \cdots \to & P_1 & \to & P_0 & \to & A & \to 0 \\ & \downarrow \theta_1 & & \downarrow \theta_0 & & \downarrow f & \\ \cdots \to & Q_1 & \to & Q_0 & \to & B & \to 0 \end{array}$$

$\exists\, \theta : P_\cdot \to Q_\cdot$ st θ induces f on $H_0(P_\cdot) \to H_0(Q_\cdot)$

Then $L_iF(f)$ is just the map from $H_i(FP_\cdot) \to H_i(FQ_\cdot)$ induced by $F(\theta) : FP_\cdot \to FQ_\cdot$. Check this is well-defined.

We saw L_iFA is independent of choice of projective resolution. We also have

Theorem 1: $L_0F = F$

Proof: $\exists\, \varphi : P_0 \twoheadrightarrow A$.

$$\cdots P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi} A \to 0$$

F is right exact so if $K = \ker(P_0 \overset{\varphi}{\twoheadrightarrow} A)$ $\quad 0 \to K \to P_0 \to A \to 0$

$\Rightarrow FK \to FP_0 \to FA \to 0$ exact.

In fact $P_1 \to P_0 \to A \to 0$ is exact $\quad FP_1 \to FP_0 \to FA \to 0$ exact

What is $L_0F(A)$? $\quad P_1 \to P_0 \to 0 \to \cdots \quad$ Apply F

$$FP_1 \to FP_0 \to 0 \to \cdots$$

& $L_0F = 0^{th}$ homology $\quad FP_0 / \mathrm{Im}(FP_1)$

But $\mathrm{Im}(FP_1) = \ker(FP_0 \to FA)$

Mitchell's embedding allows us to work in R-Mod

$$FP_0 \xrightarrow{\varphi} A \to 0$$

$$\underset{\substack{\| \\ L_0FA}}{FP_0 / \ker\varphi} \cong A$$

$\boxtimes$

Theorem 2: If A is projective then $L_iFA = 0 \;\; \forall i \geq 1$

Proof: Notice $0 \to 0 \to A \xrightarrow{id} A \to 0$ is a projective resolution of A

Apply F: $\quad \cdots \to F0 \to F0 \to \cdots \to F0 \to FA \to F0 \to \cdots$

$\because F0 = 0 \quad$ We get

$$0 \to 0 \to \cdots \to 0 \to FA \to 0 \to$$

So $H_i(FP_\cdot) = 0 \;\; \forall i \geq 1$ where $P_\cdot : \cdots \to 0 \to 0 \to A \to 0 \to \cdots$

So $L_iFA = 0 \;\; \forall i > 0$

Theorem 3: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short-exact sequence then $\exists$ a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_2"] & L_1FA \arrow[r, "L_1Ff"] & L_1FB \arrow[r, "L_1Fg"] & L_1FC \arrow[r, "\delta_1"] & \underset{L_0FA}{FA} \arrow[r, "Ff"] & \underset{L_0FB}{FB} \arrow[r, "Fg"] & \underset{L_0FC}{FC} \arrow[r] & 0
\end{tikzcd}$$

with $\delta_i : L_iFC \to L_{i-1}FA$.

Idea:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\theta_2"] & P_1 \arrow[r] \arrow[d, "\theta_1"] & P_0 \arrow[r] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & U_2 \arrow[r] \arrow[d, "\tau_2"] & U_1 \arrow[r] \arrow[d, "\tau_1"] & U_0 \arrow[r] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Like to find $U_\cdot \to B \to 0$ st $0 \to P_\cdot \xrightarrow{\theta} U_\cdot \xrightarrow{\tau} Q_\cdot \to 0$ exact.
Then claim $0 \to FP_i \to FU_i \to FQ_i \to 0$ is exact $\forall i$.

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Remark: $F : \mathcal{C} \to \mathcal{D}$ abelian categories, F additive $\Rightarrow F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$.

$$\begin{tikzcd}
0_{\mathcal{C}} \arrow[r, "0"', "\mathrm{id}"] \arrow[d] & 0_{\mathcal{C}} \arrow[d] \\
F(0_{\mathcal{C}}) \arrow[r, "\mathrm{id}"] & F(0_{\mathcal{C}})
\end{tikzcd}$$

Show $F(0_{\mathcal{C}})$ is initial and terminal

Proof (of thm 3): We reduced this to two claims:

1) Given projective resolutions

$$P_\cdot \to A \to 0$$
$$Q_\cdot \to C \to 0$$

$\exists$ a projective resolution $U_\cdot \to B \to 0$ & chain maps $\theta : P_\cdot \to U_\cdot$, $\tau : U_\cdot \to Q_\cdot$

st $0 \to P_\cdot \xrightarrow{\theta} U_\cdot \xrightarrow{\tau} Q_\cdot \to 0$ is exact & θ induces $f : A \to B$ on $H_0(P_\cdot) \xrightarrow{f} H_0(U_\cdot)$ & τ induces $g : B \to C$ on $H_0(U_\cdot) \to H_0(Q_\cdot)$

2) $0 \to P_0 \to U_0 \to Q_0 \to 0$ exact, P_0, U_0, Q_0 proj.

$\Rightarrow 0 \to FP_0 \to FU_0 \to FQ_0 \to 0$

Verification of claim 1:

$\exists\, h_0 : Q_0 \to B$ st $g \circ h_0 = \psi_0$

Let $k_0 : P_0 \to B$, $k_0 = f \circ \varphi_0$

$$\begin{tikzcd} & Q_0 \arrow[dl, "h_0"'] \arrow[d, "\psi_0"] \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

$\psi_0 : Q_0 \to C$

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] \\ p \in & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\theta_0 = i"] & & A \arrow[d, "f"] & \ni \varphi_0(p) \\ (p,0) & P_0 \oplus Q_0 \arrow[r, "h_0 + k_0"] \arrow[d, "\tau_0 = \pi"] & & B \arrow[d, "g"] & h_0(q) + k_0(p) \\ q & Q_0 \arrow[r, "\psi_0"] \arrow[d] & & C & g \circ h_0(q) \\ & 0 \end{tikzcd}$$

(p,q)

$g \circ f \circ \varphi_0 = 0$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ P_1 \arrow[r, "\varphi_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ U_1 \arrow[r] & U_0 \arrow[r, "\chi_0 = h_0 + k_0"] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\ Q_1 \arrow[r, "\psi_1"] & Q_0 \arrow[r, "\psi_0"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 \end{tikzcd}$$

$$\begin{tikzcd} P_1 \arrow[r, "\varphi_1"] \arrow[d, "i"] \arrow[dr, dashed, "k_1"] & P_0 \arrow[d, "\theta_0"] \\ P_1 \oplus Q_1 \arrow[r, "k_1 + h_1"] \arrow[d, "\pi"] & \ker(U_0 \xrightarrow{\chi_0} B) \arrow[d, "\tau_0"] \\ Q_1 \arrow[r] \arrow[ur, dashed, "h_1"] & \ker(Q_0 \xrightarrow{\psi_0} C) \arrow[r] \arrow[d] & 0 \\ & 0 \end{tikzcd}$$

continuing in this way gives the result. (ie gives claim 1)

Verification of Claim 2:

It is sufficient to show that if $0 \to P \to U \to Q \to 0$ is a short exact sequence of proj. ob then

$$0 \to FP \to FU \to FQ \to 0 \text{ is exact}$$

Why?

$$\begin{tikzcd} & & & Q \text{ proj} \arrow[dl, "s"'] \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] & P \arrow[r, "\theta"] & U \arrow[l, bend right, "t"'] \arrow[r, "\tau"] & Q \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & U \arrow[dl, "t"'] \arrow[d, "\mathrm{id} - s \circ \tau"] \\ P \arrow[r, "\theta"] & \mathrm{Im}\,\theta = \ker(\tau) \arrow[r] & 0 \end{tikzcd}$$

Apply F, F is right exact.

There was a lot of confusion but apparently it's easy.

Theorem 4:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

exact rows & commutes

$$\Rightarrow \begin{tikzcd} L_{i+1}FC \arrow[r, "\delta_{i+1}"] \arrow[d, "L_{i+1}F\gamma"] & L_iFA \arrow[d, "L_iF\alpha"] \\ L_{i+1}FC' \arrow[r, "\delta'_{i+1}"] & L_iFA' \end{tikzcd}$$

commutes

Remark: Similarly, if $G: \mathcal{C} \to \mathcal{D}$ is additive & left-exact. If $C \in Ob(\mathcal{C})$, If $\mathcal{C}$ has enough inj.

$$0 \to C \to I^0 \to I^1 \to I^2 \to \cdots$$

Apply G $\quad 0 \to GC \to GI^0 \to GI^1 \to \cdots$

$$0 \to GI^0 \to GI^1 \to \cdots \quad \text{chain complex}$$

$$H^i(GI^\cdot) =: R^iG(C)$$

We have 1) $R^0G = G$

2) C inj $\Rightarrow R^iG(C) = 0 \quad \forall i > 0$

3) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

$$0 \to GA \to GB \to GC \xrightarrow{\delta^0} R^1GA \to R^1GB \to R^1GC \to \cdots$$

$$\begin{tikzcd}
41) \quad 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r] & C' &
\end{tikzcd}$$

$$\begin{tikzcd}
R^iGC \arrow[r, "\delta^i"] \arrow[d, "R^iG\gamma"] & R^{i+1}GA \arrow[d, "R^{i+1}G\alpha"] \\
R^iGC' \arrow[r, "(\delta')^i"] & R^{i+1}GA'
\end{tikzcd}$$

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Remark: Notice thms 1-4 allow us to recover L_iFA $\forall i \geq 0$ $\forall$ objects A

$L_0FA = FA$ ✓

If we know L_iFA $\forall i < n$

recursive procedure for computing left-derived functors

$$0 \to \overset{\text{Kernel}}{K} \to \underset{\text{projective}}{P} \to A \to 0$$

$$\to FK \to FP \to FA \to 0$$

$$\to L_1FK \to 0 \to L_1FA \to$$

$$\to L_2FK \to 0 \to L_2FA \to \qquad \because L_iFP = 0 \ \forall i > 0$$

$L_2FA \cong L_1FK$, $L_3FA \cong L_2FK$

$$* \ 0 \to L_1FA \to FK \to FP \to FA \to 0$$

So knowing L_iFK gives us $L_{i+1}FA$ $\forall i \geq 1$ & we can get L_1FA from $(*)$