Complexes

$\mathcal{A}$ = abelian category. We can always work in R-Mod (Mitchell embedding theorem).

Def| A chain complex $C_\cdot$ is a family $\{C_n\}_n$, $C_n \in Ob(\mathcal{A})$ & morphisms $d_n : C_n \to C_{n-1}$ st $d_{n-1} \circ d_n = 0$. We call the d_n the differentials of $C_\cdot$
The kernel of d_n

n-cycles of $C_\cdot$ = $Z_n(C_\cdot) = \ker(d_n) \subseteq C_n$

n-boundaries of $C_\cdot$ = $B_n(C_\cdot) = \mathrm{Im}(d_{n+1}) \subseteq C_n$

So $(0) \subseteq B_n(C_\cdot) \subseteq Z_n(C_\cdot) \subseteq C_n$

Define $H_n(C_\cdot) = Z_n(C_\cdot)/B_n(C_\cdot)$ called the n^{th} homology group of $C_\cdot$

Remark: $H_n(C_\cdot) = (0) \iff C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$ exact at C_n.

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Dually we define a cochain complex $C^\cdot$ to be a family $(C^n ; n \in \mathbb{Z})$ and morphisms $d^n : C^n \to C^{n+1}$ st $d^{n+1} \circ d^n = 0$ $\forall n \in \mathbb{Z}$. Define $Z^n(C^\cdot) = \ker(d^n) \subseteq C^n$ to be n-cocycles, $B^n(C^\cdot) = \mathrm{im}(d^{n-1}) \subseteq C^n$ to be n-coboundaries, $H^n(C^\cdot) = Z^n(C)/B^n(C)$ to be the n^{th} cohomology group of $C^\cdot$

Remark: $(C_n ; n \in \mathbb{Z})$ is a chain complex $\iff B^n = C_{-n}$ with $d^n = d_{-n}$ is a cochain complex

Ex [de Rham complex] Suppose $\varphi : R \to A$ is an R-algebra. Recall the Kähler differentials are $\Omega_{A/R}$, the free A-module generated by symbols da, $a \in A$, modulo the relations

$d(a + rb) - da - rdb$, $d(ab) - adb + bda$, $d1 = 0$ $a, b \in A$, $r \in R$

Now define

$$\Omega^i_{A/R} = \Lambda^i \Omega_{A/R} = \bigotimes_{j=1}^{i} \Omega_{A/R} \Big/ \langle a_1 \otimes \cdots \otimes a_i - \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} \rangle$$

We also take $\Omega^i_{A/R} = 0$ for $i < 0$. Given $m_1, \ldots, m_i \in \Omega_{A/R}$ we let $m_1 \wedge \cdots \wedge m_i$ denote the image of $m_1 \otimes \cdots \otimes m_i$ in $\Lambda^i \Omega_{A/R}$. Note:

- $\Omega^0_{A/R} = A$

- $\Omega^1_{A/R} = \Omega_{A/R}$
- have map $d: A \to \Omega_{A/R}$, $a \mapsto da$; call this $d^0: \Omega^0_{A/R} \to \Omega^1_{A/R}$
- have another map $d^1: \Omega^1_{A/R} \to \Omega^2_{A/R}$ given by $d^1(adb) = da \wedge db$ (in particular, $d^1 \circ d^0 = 0$)
- in general, these yield a map $d^n: \Omega^n_{A/R} \to \Omega^{n+1}_{A/R}$ satisfying $\forall \omega \in \Omega^i_{A/R}, \eta \in \Omega^{n-i}_{A/R}$ $d^n(\omega \wedge \eta) = d^i \omega \wedge \eta + (-1)^i \omega \wedge d^{n-i} \eta$

In particular, we take
$d^n(\omega_1 \wedge \cdots \omega_n) = (d^1 \omega_1 \wedge \cdots \wedge \omega_n) - (\omega_1 \wedge d^1 \omega_2 \wedge \cdots) + (\omega_1 \wedge \omega_2 \wedge d^1 \omega_3 \wedge \cdots) - \cdots$

- In particular, for $\omega \in \Omega^{n-1}_{A/R}$, $\eta \in \Omega^1_{A/R}$ we have

$$
\begin{aligned}
(d^{n+1} \circ d^n)(\omega \wedge \eta) &= d^{n+1}(d^{n-1}\omega \wedge \eta + (-1)^{n-1} \omega \wedge d\eta) \\
&= d^{n+1}(d^{n-1}\omega \wedge \eta) + (-1)^{n-1} d^{n+1}(\omega \wedge d\eta) \\
&= d(d^{n-1}(\omega)) \wedge \eta + (-1)^n d^{n-1}\omega \wedge d^1\eta + (-1)^{n-1} d^{n-1}\omega \wedge d^1\eta + (-1)^{n-1}(-1)^{n-1} \omega \wedge d^2(d\eta) \\
&= 0
\end{aligned}
$$

by an inductive argument

Exercise: Suppose k is a field of characteristic 0; let $A = k[x_1, \ldots, x_n]$. Then
$0 \to k \to \Omega^0_{A/k} \to \Omega^1_{A/k} \to \cdots \to \Omega^n_{A/k} \to 0$ is exact

Def) Let $C_\cdot$ and $C'_\cdot$ be two chain complexes; say $C_\cdot = (C_n, d_n)$ and $C'_\cdot = (C'_n, d'_n)$. A morphism of chain complexes is a collection of maps $f_n: C_n \to C'_n$ s.t. the following commutes:

$$
\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_n \arrow[r, "d'_n"] & C'_{n-1}
\end{tikzcd}
$$

Thus if $\mathcal{C}$ is an abelian cat. then we can set $Ch(\mathcal{C})$ to be the category of chain complexes in $\mathcal{C}$. Similarly, we define $Co\text{-}Ch(\mathcal{C})$, the category of cochain complexes in $\mathcal{C}$.

In fact, $Ch(\mathcal{C})$ and $Co\text{-}Ch(\mathcal{C})$ are ab. cat. The only non-trivial part is checking that $\ker(f)$ and $\operatorname{coker}(f)$ are objects in $Ch(\mathcal{C})$ for $f: C_\cdot \to C'_\cdot$. One can assume $\mathcal{C} = R\text{-Mod}$ by Mitchell's embedding thm. Note then that the following diagram commutes:

$$\begin{tikzcd}
\ker(f_n) \arrow[r, "d_n|_{\ker(f_n)}"] \arrow[d] & \ker(f_{n-1}) \arrow[d] \\
C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_n \arrow[r, "d'_n"] \arrow[d] & C'_{n-1} \arrow[d] \\
\operatorname{coker}(f_n) \arrow[r, "\overline{d'_n}"] & \operatorname{coker}(f_{n-1})
\end{tikzcd}$$

since if $x' = x'' + f_n(u)$ in C'_n then

$$d'_n(x') = d'_n(x'') + (d'_n \circ f_n)(u) = d'_n(x'') + (f_{n-1} \circ d_n)(u)$$

and $d'_n(x') = d'_n(x'')$ in $\operatorname{coker}(f_{n-1})$. One also checks that monomorphisms and epimorphisms are normal; hence $\mathrm{Ch}(\mathcal{C})$ is an ab cat.

Remark: One can show that a morphism $C_\bullet \to C'_\bullet$ takes $Z_n(C_\bullet)$ to $Z_n(C'_\bullet)$ and $B_n(C_\bullet)$ to $B_n(C'_\bullet)$; in particular we get a map $H_n(C_\bullet)$ to $H_n(C'_\bullet)$.

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$u : C_\bullet \to D_\bullet$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[d, "u_{n-1}"] & \\
D_{n+1} \arrow[r, "d'_{n+1}"] & D_n \arrow[r, "d'_n"] & D_{n-1} \arrow[r] & {}
\end{tikzcd}$$

u induces a morphism $u : H_n(C_\bullet) \to H_n(D_\bullet)$

$$x \in Z_n(C_\bullet) \xrightarrow{u_n} Z_n(D_\bullet)$$
$$x \in B_n(C_\bullet) \longrightarrow B_n(D_\bullet)$$

A morphism $u : C_\bullet \to D_\bullet$ is called a quasi-isomorphism if $\forall n$ the induced maps $H_n(C_\bullet) \to H_n(D_\bullet)$ are all isomorphisms.

Proposition. TFAE:

1) The chain complex $C_\bullet$ is exact at each d_n
2) $H_n(C_\bullet) = 0 \quad \forall n$
3) $C_\bullet$ is quasi-isomorphic to the zero chain complex

A chain complex $C_\bullet$ is bounded if $C_n = 0 \ \forall n \gg 0 \ \& \ \forall n \ll 0 \longrightarrow \mathrm{Ch}_b(\mathcal{C})$

- bounded below if $C_n = 0 \ \forall n \ll 0 \longrightarrow \mathrm{Ch}_-(\mathcal{C})$
- bounded above if $C_n = 0 \ \forall n \gg 0 \longrightarrow \mathrm{Ch}_+(\mathcal{C})$

$$\begin{tikzcd}[column sep=small]
& Ch(\mathcal{C}) & \\
Ch_-(\mathcal{C}) \arrow[ur, phantom, "\subseteq"] & \text{IU} & Ch_+(\mathcal{C}) \arrow[ul, phantom, "\supseteq"] \\
& Ch_b(\mathcal{C}) \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

full sub cat

Similarly, we have $Co\text{-}Ch^b$, $Co\text{-}Ch^-$, $Co\text{-}Ch^+$

Remark: Since $Ch(\mathcal{C})$ (resp. $Co\text{-}Ch(\mathcal{C})$) is an ab cat, it makes sense to talk about short exact sequences of chain complexes

$$0 \longrightarrow A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \longrightarrow 0$$

$$\begin{tikzcd}
& \ker(f_n) \arrow[r, "d_n|"] \arrow[d, "i_n"] \arrow[dr, phantom, "\circlearrowright"] & \ker(f_{n-1}) \arrow[d, "i_{n-1}"] & \\
\arrow[r] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_n'"] & B_{n-1} \arrow[r] &
\end{tikzcd}$$

$f: A \to B$ a monomorphism $\iff$ $\longrightarrow \ker(f_n) \longrightarrow \ker(f_{n-1}) \longrightarrow \cdots$ is the zero chain complex

$A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot$ is exact at $B_\cdot$ $\iff$

$$\begin{tikzcd}
& A_n \arrow[r, "a_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] & \\
\arrow[r] & B_n \arrow[r, "b_n"] \arrow[d, "g_n"'] & B_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
& C_n \arrow[r, "c_n"] & C_{n-1} &
\end{tikzcd}$$

$g_n \circ f_n = 0 \ \forall n$ &
$\ker(g_n)/\mathrm{Im}(f_n) = 0 \ \forall n$

Long exact sequence

If $0 \longrightarrow A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \longrightarrow 0$ is a short exact seq in $Ch(\mathcal{C})$

Then $\exists$ <u>connecting morphisms</u> $\partial_n : H_n(C) \to H_{n-1}(A)$ st

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{f} H_{n-1}(B) \xrightarrow{g} H_{n-1}(C) \longrightarrow$$

Dually, If $0 \longrightarrow A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot \longrightarrow 0$ is a s.e.s. in $Co\text{-}Ch(\mathcal{C})$

$\exists$ $\partial^n : H^n(C) \longrightarrow H^{n+1}(A)$

$$H^{n-1}(C) \xrightarrow{\partial^{n-1}} H^n(A) \xrightarrow{f} H^n(B) \longrightarrow H^n(C) \xrightarrow{\partial^n} H^{n+1}(A)$$

Key ingrediant in this proof is the snake lemma.

Snake Lemma: Suppose that $\mathcal{C}$ is an abelian cat & suppose we have a comm. diagram with exact rows. Then $\exists$ $\delta : \ker(h) \to \operatorname{coker}(f)$ st

$$\begin{tikzcd}
& \ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
& A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[d] & \\
& \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h) &
\end{tikzcd}$$

$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$ is exact

Moreover, i' is a monomorphism $\Rightarrow 0 \to \ker(f) \to \ker(g)$ is exact

p ~~is~~ an epimorphism $\Rightarrow \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0$ is exact

Proof: 1) WLOG $\mathcal{C} = R\text{-Mod}$ for some R (Mitchell's thm)

2) only hard parts are

i) finding δ

ii) Showing $\ker(g) \xrightarrow{p'|} \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \xrightarrow{\bar{i}} \operatorname{coker}(g)$ exact at

What is δ? Let $x \in \ker(h) \subseteq C'$. Take y st $p'(y) = x$

Apply g to y; $g(y) \in B$. Claim $\exists\, a \in A$ st $i(a) = g(y)$

Define $\delta(x) = a + \operatorname{Im}(f) \in \operatorname{coker}(f)$

" $\delta = \overline{i^{-1} \circ g \circ (p')^{-1}}$ "

Why is this well-defined/defined?

$$\begin{tikzcd}
& A' \arrow[r, "i'"] & B' \ni y_1, y_2 \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C &
\end{tikzcd}$$

$g(y_1), g(y_2)$ are both in $\ker(p) = \operatorname{im}(i)$ so $\exists!\, a_i \in A$ st $i(a_i) = g(y_i)$, $i = 1, 2$

Claim: $i(a_1) + \operatorname{Im}(f) = i(a_2) + \operatorname{Im}(f)$ just kind of do it...

Want now to show (ii) above.

ie need to show 1) $\mathrm{Im}(p'|_{\ker(g)}) = \ker(\delta)$

2) $\mathrm{Im}(\delta) = \ker(\bar{i})$

In 1, we showed $\subseteq$ so we must show that $x \in \ker(\delta) \Rightarrow \exists y \in \ker(g)$ st $x = p'(y)$

$x \in \ker(\delta) \Rightarrow i^{-1} \circ g \circ (p')^{-1}(x) = 0 \Rightarrow i^{-1} \circ g \circ (p')^{-1}(x) \in \mathrm{Im}(f)$

ie $i^{-1} \circ g \circ \underbrace{(p')^{-1}(x)}_{\text{pick any preimage } z} = f(a)$ some a

$\Rightarrow g \circ (p')^{-1}(x) = i \circ f(a) = g \circ i'(a) \Rightarrow g(z) = g \circ i'(a)$

So $z - i'(a) \in \ker(g)$. What is p' of this? $p'(z - i'(a)) = p'(z) - p' \circ i'(a) = x$

So $x \in p'(\ker(g)) \Rightarrow \mathrm{Im}(p'|_{\ker(g)}) \supseteq \ker(\delta)$

2) First, to show $\mathrm{Im}(\delta) \subseteq \ker(\bar{i})$ look at $\bar{i} \circ \delta(x) =$

$\bar{i}(\overline{i^{-1} \circ g \circ (p')^{-1}(x)}) = \bar{i}(i^{-1} \circ g \circ (p')^{-1}(x) + \mathrm{Im}(f))$

$= i(i^{-1} \circ g \circ (p')^{-1})^{-1}(x) + \mathrm{Im}(f)) + \mathrm{Im}(g)$

$= \underbrace{g \circ (p')^{-1}(x)}_{\in\,\mathrm{Im}(g)} + \underbrace{\mathrm{Im}(i \circ f)}_{=\,\mathrm{Im}(g \circ i')} + \mathrm{Im}(g) = 0 + \mathrm{Im}(g) = \bar{0}$

To do the reverse inclusion, let $x \in \ker(\bar{i})$; we'll show $x \in \mathrm{Im}(\delta)$

$x \in \mathrm{coker}(\delta)$ $\quad x = x_0 + \mathrm{Im}(f)$, $x_0 \in A$

$\bar{i}(x) = 0 \Rightarrow i(x_0) + \mathrm{Im}(g) = 0 + \mathrm{Im}(g) \Rightarrow i(x_0) = g(u)$, $u \in B'$

$\Rightarrow x_0 = i^{-1} \circ g(u)$

Notice if we know $\underbrace{p'(u)}_{=\,t} \in \ker(h)$ then $\delta(t) = \overline{i^{-1} \circ g \circ (p')^{-1}(t)}$

$= \overline{i^{-1} \circ g(u)}$

So it suffices to show $p'(u) \in \ker(h)$; $\quad = \bar{x}_0 = x$ So we'd be done.

ie $h \circ p'(u) = 0$ But $h \circ p'(u) = p \circ g(u) = p \circ i(x_0) = 0$ ✓ $\blacksquare$

Recall goal

if $A_\bullet, B_\bullet, C_\bullet$ $\quad 0 \to A_n \xrightarrow{f} B_n \xrightarrow{g} C_n \to 0$ exact

then

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$$

long exact sequence

Proof:

$$\begin{tikzcd}
 & 0 \arrow[r] & Z_n(A_\bullet) \arrow[r] & Z_n(B_\bullet) \arrow[r] & Z_n(C_\bullet) \arrow[dddlll, dotted, rounded corners] & \\
 & 0 \arrow[r] & A_n \arrow[r] \arrow[d, "a_n"] & B_n \arrow[r] \arrow[d, "b_n"] & C_n \arrow[r] \arrow[d, "c_n"] & 0 \\
 & & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & A_{n-1}/\mathrm{Im}(a_n) \arrow[r] & B_{n-1}/\mathrm{Im}(b_n) \arrow[r] & C_{n-1}/\mathrm{Im}(c_n) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma $0 \to Z_n(A_\bullet) \to Z_n(B_\bullet) \to Z_n(C_\bullet)$ &

$A_{n-1}/\mathrm{Im}(a_n) \to B_{n-1}/\mathrm{Im}(b_n) \to C_{n-1}/\mathrm{Im}(c_n)$ are exact $\forall n \in \mathbb{Z}$

Claim: $a_n : A_n/\mathrm{Im}(a_{n+1}) \to Z_{n-1}(A) \subseteq A_{n-1}$

If $x \in A_n$, $a_{n-1} \circ a_n(x) = 0 \Rightarrow a_n(x) \in \ker(a_{n-1}) = Z_{n-1}(A_\bullet)$

So the snake lemma gives

$$\begin{tikzcd}
& \ker(a_n) \arrow[r] \arrow[d] & \ker(b_n) \arrow[r] \arrow[d] & \ker(c_n) \arrow[d] \arrow[ddll, "\partial_n", bend left, swap] & \\
& A_n/\mathrm{Im}(a_{n+1}) \arrow[r] \arrow[d] & B_n/\mathrm{Im}(b_{n+1}) \arrow[r] \arrow[d] & C_n/\mathrm{Im}(c_{n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
& \mathrm{coker}(a_n) \arrow[r] & \mathrm{coker}(b_n) \arrow[r] & \mathrm{coker}(c_n) &
\end{tikzcd}$$

Claim: $\ker(a_n) = H_n(A_\bullet)$

$\ker(a_n) = \{x + \mathrm{Im}(a_{n+1}) ; a_n(x) = 0\} = Z_n(A_\bullet)/B_n(A_n)$

$\mathrm{coker}(a_n) = Z_{n-1}(A_n)/\mathrm{Im}(\cancel{\partial}\, a_n) = Z_{n-1}(A_\bullet)/B_{n-1}(A_\bullet) = H_{n-1}(A_\bullet)$

So we get $H_n(A_\bullet) \xrightarrow{f} H_n(B_\bullet) \xrightarrow{g} H_n(C_\bullet) \xrightarrow{\partial_n} H_{n-1}(A_\bullet) \xrightarrow{f} H_{n-1}(B_\bullet) \xrightarrow{g} H_{n-1}(C_\bullet)$ is exact $\forall n$

Putting these together gives the long exact sequence. $\blacksquare$

Homotopies of complexes

Let $\alpha, \beta : A_\bullet \to B_\bullet$ be two maps of chain complexes

$A_\bullet = (A_n, a_n)$, $B_\bullet = (B_n, b_n)$

Def| We say that α is homotopic to β (α homotopy equivalent to β), written $\alpha \sim \beta$, if $\forall n \in \mathbb{Z}$ $\exists h_{n-1} : A_{n-1} \to B_n$

$h_{n-1} \in \mathrm{Hom}_{\mathcal{C}}(A_{n-1}, B_n)$ st

$$\alpha_n - \beta_n = h_{n-1} \circ a_n + b_{n+1} \circ h_n$$

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$$\begin{tikzcd}
\arrow[r] & A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, shift left] \arrow[d, shift right] & A_n \arrow[r, "a_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\alpha_n"'] \arrow[d, shift left, "\beta_n"] & A_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "\alpha_{n-1}"'] \arrow[d, shift left, "\beta_{n-1}"] & {} \\
\arrow[r] & B_{n+1} \arrow[r, "b_{n+1}"'] & B_n \arrow[r, "b_n"'] & B_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Remark: $\sim$ is indeed an equivalence relation (check)

Proposition: If $\alpha, \beta : (A_n, a_n) \to (B_n, b_n)$ are homotopy equivalent then α & β induce the same maps $H_n(A_\cdot) \to H_n(B_\cdot)$ $\forall n \in \mathbb{Z}$

Proof: It suffices to show that if $(\alpha - \beta =) \gamma \sim 0$ then γ induces the 0-map $H_n(A_\cdot) \to H_n(B_\cdot)$.

Suppose $\gamma_n = h_{n-1} \circ a_n + b_{n+1} \circ h_n$

$H_n(A_\cdot) = Z_n(A_\cdot)/B_n(A_\cdot) = \ker(a_n)/\mathrm{Im}(a_{n+1})$

$H_n(B_\cdot) = \ker(b_n)/\mathrm{Im}(b_{n+1})$

$$\gamma : \ker(a_n)/\mathrm{Im}(a_{n+1}) \to \ker(b_n)/\mathrm{Im}(b_{n+1})$$
$$x + \mathrm{Im}(a_{n+1}) \mapsto \gamma_n(x) + \mathrm{Im}(b_{n+1})$$

So to show that γ induces the 0-map, we must show
$$\gamma_n(\ker(a_n)) \subseteq \mathrm{Im}(b_{n+1}).$$

Take $x \in A_n$ st $a_n(x) = 0$
$$\gamma_n(x) = \underbrace{h_{n-1}(\underbrace{a_n(x)}_{0})}_{0} + b_{n+1} \circ h_n(x) \in \mathrm{Im}(b_{n+1}) \checkmark$$
$\blacksquare$

Key Proposition: Let
$$F_\cdot : \to F_i \xrightarrow{\varphi_i} F_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} F_0 \to 0 \to \cdots$$
$$G_\cdot : \to G_i \xrightarrow{\psi_i} G_{i-1} \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_1} G_0 \to 0 \to \cdots$$
be two chain complexes in an abelian category $\mathcal{C}$ (work in R-Mod) & suppose $\forall i$, F_i, G_i are projective objects. In addition, assume

1) $M = \mathrm{coker}\,\varphi_1 = H_0(F_\cdot)$
2) $N = \mathrm{coker}\,\psi_1 = H_0(G_\cdot)$
3) $H_i(G_\cdot) = 0$ $\forall i > 0$

Then if $\beta : M \to N \Rightarrow \beta$ is a map induced on $\underset{M}{H_0(F_\cdot)} \to \underset{N}{H_0(G_\cdot)}$ by a chain map $\alpha : F_\cdot \to G_\cdot$. Moreover α is determined by β up to homotopy equivalence.

Proof: By induction. <u>Existence:</u>

$$\begin{tikzcd}
F_2 \arrow[r, "\varphi_2"] \arrow[d, dashed, "\alpha_2"] & F_1 \arrow[r, "\varphi_1"] \arrow[d, dashed, "\alpha_1"] & F_0 \arrow[r, "\pi_F"] \arrow[d, dashed, "\alpha_0"] \arrow[dr, "\beta\circ\pi_F"] & M \arrow[r] \arrow[d, "\beta"] & 0 \\
G_2 \arrow[r, "\psi_2"] & G_1 \arrow[r, "\psi_1"] & G_0 \arrow[r, "\pi_G"'] & N \arrow[r] & 0
\end{tikzcd}$$

$\because F_0$ is proj. $\exists\, \alpha_0 : F_0 \to G_0$ st $\pi_G \circ \alpha_0 = \beta \circ \pi_F$

Now $\alpha_0\circ\varphi_1 : F_1 \to G_0$ we have $\mathrm{Im}(\alpha_0\circ\varphi_1) \subseteq \ker(G_0 \xrightarrow{\pi_G} N)$

Why? $\pi_0\circ\alpha_0\circ\varphi_1 = \beta\circ\underbrace{\pi_F\circ\varphi_1}_{0 \text{ by exactness}} = 0$

So $\because F_1$ is projective, $\exists\, \alpha_1 : F_1 \to G_1$

$\psi_1\circ\alpha_1 = \alpha_0\circ\varphi_1$

$$\begin{tikzcd} & F_1 \arrow[dl, dashed, "\alpha_1"'] \arrow[d] & \\ G_1 \arrow[r] & \mathrm{Im}(\psi_1) \arrow[r] & 0 \end{tikzcd}$$

Continuing in this manner gives a chain map $\alpha : F_\cdot \to G_\cdot$.

Moreover,

$$\alpha_0 : \overset{M}{\overset{\shortparallel}{F_0/\mathrm{Im}\,\varphi_1}} \longrightarrow \overset{N}{\overset{\shortparallel}{G_0/\mathrm{Im}\,\psi_1}}$$

$\alpha_0(x+\mathrm{Im}\,\varphi_1)$, $x\in F_0$ $= \alpha_0(x) + \mathrm{Im}(\alpha_0\circ\varphi_1) = \underset{\overset{?}{\shortparallel}\ \beta(x)+\mathrm{Im}(\psi_1)}{\alpha_0(x) + \mathrm{Im}(\psi_1\circ\alpha_1)}$

Now $\pi_G\circ\alpha_0 = \beta\circ\pi_F \Rightarrow \pi_G(\alpha_0(x)) = \beta(x+\mathrm{Im}(\varphi_1))$

$\Rightarrow \alpha_0(x) + \mathrm{Im}(\psi_1) = \beta(x) + \mathrm{Im}\,\psi_1$

So β is induced by the chain map α.

<u>Uniqueness</u>: Suppose that $\alpha, \alpha' : F_\cdot \to G_\cdot$ both induce β. Must show $\alpha \sim \alpha'$. Reduction to showing $\gamma : F_\cdot \to G_\cdot$ induces $0 : M \to N \Rightarrow \gamma \sim 0$.

So we assume $\beta : M \to N$ is the zero map

$$\begin{tikzcd} F_1 \arrow[r, "\varphi_1"] \arrow[d, "\gamma_1"] & F_0 \arrow[r, "\pi_F"] \arrow[d, "\gamma_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ G_1 \arrow[r, "\psi_1"'] & G_0 \arrow[r, "\pi_G"'] & N \arrow[r] & 0 \end{tikzcd}$$

Suppose $\gamma : F_\cdot \to G_\cdot$ induces 0.

Claim: $\mathrm{Im}(\gamma_0) \subseteq \mathrm{Im}\,\psi_1 = \ker(\pi_G)$

Pf: $\pi_G\circ\gamma_0 = 0\circ\pi_F = 0 \Rightarrow \uparrow$

So $\gamma_1\circ h_0 = \gamma_0$.

Goal: produce $h_1 : F_1 \to G_2$ st $\psi_2\circ h_1 + h_0\circ\varphi_1 = \gamma_1$ 2016 03 15

We know $\gamma_0 = \psi_1\circ h_0$

So $\psi_1\circ h_0\circ\varphi_1 - \psi_1\circ\gamma_1 = \psi_1\circ(h_0\circ\varphi_1 - \gamma_1)$

On the other hand $\underbrace{(\psi_1\circ h_0)}\circ\varphi_1 - \psi_1\circ\gamma_1$

$\gamma_0\circ\varphi_1 - \psi_1\circ\gamma_1 = 0 \because \square$ commutes

$= \gamma_0\circ\varphi_1 - \psi_1\circ\gamma_1 = 0$

So $\psi_1\circ(h_0\circ\varphi_1 - \gamma_1) = 0 \Rightarrow \mathrm{Im}(h_0\circ\varphi_1 - \gamma_1) \subseteq \ker\psi_1 = \mathrm{Im}(\psi_2)$

$$\begin{tikzcd} & F_1 \arrow[dl, "h_1"'] \arrow[d, "-h_0\circ\varphi_1+\gamma_1"] & \\ G_2 \arrow[r, "\psi_2"] & \mathrm{Im}(\psi_2) \arrow[r] & 0 \end{tikzcd}$$

$\because F$ is proj, $\exists\, h_1 : F_1 \to G_2$ st

$\psi_2\circ h_1 = -h_0\circ\varphi_1 + \gamma_1$

$\Rightarrow \gamma_1 = \psi_2\circ h_1 + h_0\circ\varphi_1$

Continuing in this manner, we see that $\gamma \sim 0$.

Projective Resolution

$\mathcal{C}$ be an abelian cat with enough projectives (resp. enough injectives)

Then if we have enough proj., we can (find) a projective resolution of C

$$\cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow C \longrightarrow 0$$

i.e. $H_i(P_\cdot) = 0 \quad \forall i > 0 \quad C = \operatorname{coker}(\varphi_1)$

i.e. it is exact, P_i are proj.

Why? Work in R-Mod $\exists P_0 \overset{\varphi_0}{\twoheadrightarrow} C$, P_0 proj.

$$0 \to K_0 \to P_0 \to C \to 0$$

Let $K_0 = \ker(P_0 \to C)$ $\quad \exists P_1 \overset{\varphi_1}{\twoheadrightarrow} K_0$, P_1 proj

So $\quad P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \to 0 \quad \operatorname{Im}(\varphi_1) = K_0 = \ker(\varphi_0)$

Let $K_1 = \ker(\varphi_1)$ Keep going.

Similarly, if we (have) enough inj. we get an injective resolution of C,

$$0 \to C \to I_0 \to I_1 \to I_2 \to \cdots \quad \text{exact, } I_i \text{ inj}$$

Theorem: Let $C \in \operatorname{Ob}(\mathcal{C})$. If

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \xrightarrow{} 0$$

& $$\to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \xrightarrow{\psi_0} C \xrightarrow{} 0$$

are two proj. res. of C. Then

1) $P_\cdot : \cdots \to P_2 \cdots \longrightarrow C \xrightarrow{0} 0 \to 0 \to 0 \cdots$

$Q_\cdot : \cdots \to Q_2 \cdots \longrightarrow C \xrightarrow{0} 0 \to 0 \to 0 \cdots$

are homotopy equivalent

2) If $F: \mathcal{C} \to \mathcal{D}$ (ab cats) is an additive Functor then $\forall i$

$$H_i(FP_\cdot) \cong H_i(FQ_\cdot)$$

Remark: $\quad \to FP_1 \xrightarrow{F\varphi_1} FP_0 \xrightarrow{0} 0 \to \cdots$

$\to FQ_1 \xrightarrow{F\psi_1} FQ_0 \xrightarrow{0} 0 \to \cdots \quad$ chain complexes

$(F\varphi_i) \circ (F\varphi_{i+1}) = F(\varphi_i \circ \varphi_{i+1}) = F(0) = 0 \quad \because F$ additive So it is a chain complex

Idea:

$$\begin{tikzcd}
\arrow[r] & P_2 \arrow[r] \arrow[d, "\alpha_2"] & P_1 \arrow[r] \arrow[d, "\alpha_1"] & P_0 \arrow[r] \arrow[d, "\alpha_0"] & C \arrow[r] \arrow[d, "id"] & 0 \\
\arrow[r] & Q_2 \arrow[r] \arrow[d, "\beta_2"] & Q_1 \arrow[r] \arrow[d, "\beta_1"] & Q_0 \arrow[r] \arrow[d, "\beta_0"] & C \arrow[r] \arrow[d, "id"] & 0 \\
\arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

By our last result, $\exists\, \alpha : P_\bullet \to Q_\bullet$, $\beta : Q_\bullet \to P_\bullet$ st α induces $id : C \to C$

β " $id : C \to C$

So $\beta \circ \alpha : P_\bullet \to P_\bullet$ induces $id : C \to C$ But $id_{P_\bullet} : P_\bullet \to P_\bullet$ also induces $id : C \to C$ so $\beta \circ \alpha \sim id_{P_\bullet}$

Similarly $\alpha \circ \beta \sim id_{Q_\bullet}$

Now consider $\beta \circ \alpha : P_\bullet \to P_\bullet$

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r] \arrow[d, "\beta_2 \circ \alpha_2"] & P_1 \arrow[r] \arrow[d, "\beta_1 \circ \alpha_1"] \arrow[dl, "h_1"'] & P_0 \arrow[r] \arrow[d, "\beta_0 \circ \alpha_0"] \arrow[dl, "h_0"'] & 0 \arrow[r] \arrow[dl, "0"'] & {} \\
 & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

So $\exists\, h_i : P_i \to P_{i+1}$ st

$$\beta_i \circ \alpha_i - id_{P_i} = \varphi_{i+1} \circ h_i + h_{i-1} \circ \varphi_i$$

Apply F everywhere!

$$F(\beta_i) \circ F(\alpha_i) - id_{FP_i} = F(\varphi_{i+1}) \circ F(h_i) + F(h_{i-1}) F(\varphi_i)$$

So $F(h_i) : FP_i \to FP_{i+1}$

Give $F(\alpha) : FP_\bullet \to FQ_\bullet$, $F(\beta) : FQ_\bullet \to FP_\bullet$ satisfy

$F(\beta) \circ F(\alpha) \sim id_{FP_\bullet}$

Similarly $F(\alpha) \circ F(\beta) \sim id_{FQ_\bullet}$

So $F(\beta) \circ F(\alpha)$ & $id_{FP_\bullet}$ induce the same map (ie the identity map) from $H_i(FP_\bullet) \to H_i(FP_\bullet)$ $\forall i$

Similarly, $F(\alpha) \circ F(\beta)$ induces the id on $H_i(FQ_\bullet)$ $\forall i$