Loose ends

1) Grothendieck group

R a ring

We can make a group $K_0(R)$ collection of f.g. (left) proj. R-mod. up to iso.

Let A = free ab. gr. on all iso classes $[P]$, P f.g. proj.

We'll impose relations $[P_1]+[P_2]=[P_3]$ whenever $\exists$ short exact seq

$$0 \to P_1 \to P_3 \to P_2 \to 0$$
$$0 \to k^{n-1} \to k^n \to k \to 0$$
$$[k^n] = [k^{n-1}] + [k]$$

$K_0(k) \cong \mathbb{Z}$ $\quad [k^n] \mapsto n$

If R is comm. $K_0(R)$ is a ring $\quad [P]\cdot[Q] = [P \otimes Q]$

P proj $\to P \oplus H \cong R^n$ $\quad Q$ proj $\Rightarrow Q \oplus E \cong R^m$

$$R^{nm} \cong R^n \otimes_R R^m \cong (P \oplus H) \otimes (Q \oplus E) \cong (P \otimes Q) \oplus (H \otimes Q) \oplus (H \otimes E) \oplus (P \otimes E)$$

Exterior products

R - comm. ring $\quad M$ is an R-mod

$$\Lambda^i M = i^{th} \text{ exterior prod of } M$$
$$\underbrace{M \otimes_R \cdots \otimes_R M}_{i} / N$$

where N is the submodule gen by $m_1 \otimes \cdots \otimes m_i = \operatorname{sgn}(\sigma) m_{\sigma(1)} \otimes \cdots m_{\sigma(i)}$

$\Lambda^0 M = R \quad \Lambda^1 M = M, \quad \Lambda^i R^n \cong R^{\binom{n}{i}}$

Why? $e_1 - e_n$ basis for R^n. $\quad e_{i_1} \otimes \cdots \otimes e_{i_n}$ span

$= \pm e_{l_1} \otimes - - e_{l_n} \quad l_1 \le \cdots \le l_n$

duplicates die if char $\neq 2$

So can show $e_{l_1} \otimes - \otimes e_{l_i}$, $1 \le l_1 < \cdots < l_i \le n$ is a basis

In particular $\Lambda^n R^n \cong R$

If R is a noeth comm ring with spec(R) con, then if P is a proj. module of rank n $\Lambda^i P$ is proj of rank $\binom{n}{i}$

Now we let $\text{Pic}(R)$ denote the multiplicative subset of $K_0(R)$ gen by rank 1 projectives $[P],[Q] \to [P \otimes Q]$

$\text{Pic}(R)$ is called the Picard grp of R. It is a group

$$[P] \otimes [\mathrm{Hom}(P,R)] = [R] \leftarrow \text{identity}$$

We have a map

$$K_0(R) \longrightarrow \mathrm{Pic}(R)$$
$$[P] \longmapsto [\Lambda^{\mathrm{rank}(P)} P]$$

This is a homeomorphism of semigroups (under $\otimes$)

Final remark

If R is comm then if $P \oplus R^n \cong R^{n+1} \Rightarrow P \cong R$

This is an exercise

Step 1 $\Lambda^i(M \oplus N) \cong \oplus_{j=0}^{i} \Lambda^j(M) \otimes \Lambda^{i-j}(N)$

Step 2 $R^{n+1} \cong R^n \oplus P$

$$R = R^{\binom{n+1}{n+1}} \cong \Lambda^{n+1} R^{n+1} \cong \Lambda^{n+1}(R^n \oplus P) \cong \oplus_{j=0}^{n+1} \Lambda^j R^n \otimes \Lambda^{n+1-j}(P)$$

Step 3 Show that $\because$ P has rank 1, $\Lambda^j P = (0)$ for $j > 1$ & $\Lambda^{n+1} R^n = (0)$

Only non-zero term when $j = n$: $R \cong \Lambda^n R^n \otimes \Lambda^1 P \cong R \otimes P = P$

Injective Modules

Notion is dual to projectives

A abelian cat, P proj. obj.: $\mathrm{Hom}(P,-)$ is exact

I injective obj: $\mathrm{Hom}(-,I)$ is exact

i.e. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ exact

$$\Downarrow$$

$$0 \longrightarrow \mathrm{Hom}(C,I) \longrightarrow \mathrm{Hom}(B,I) \longrightarrow \mathrm{Hom}(A,I) \longrightarrow 0 \text{ exact}$$
$$\psi \longmapsto \psi \circ g \qquad \varphi \longmapsto \varphi \circ f$$

So this is equivalent (one can check) to 2016 02 26

$$0 \to A \xrightarrow{f} B \text{ exact} \Rightarrow \mathrm{Hom}(B,I) \xrightarrow{-\circ f} \mathrm{Hom}(A,I) \to 0 \text{ exact}$$
$$\psi \longmapsto \psi \circ f$$

i.e.,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "\exists \bar{h}"] \\ & I & \end{tikzcd}$$

Lemma (Baer): Let R be a ring & let Q be a left R-module. If $\forall I \leq R$ left ideal & $h: I \to Q$ R-Mod homo $\exists \tilde{h}: R \to Q$ st $\tilde{h}|_I = h$ R-Mod hom, then Q is injective.

Proof: $$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\beta"] & M \\ & Q & \end{tikzcd}$$ f inj., WLOG f is inclusion

Consider all pairs (N', β') st $N \subseteq N' \subseteq M$, $\beta': N' \to Q$, $\beta'|_N = \beta$. We can partially order these pairs via $(N_1, \beta_1) \leq (N_2, \beta_2)$ if $N_1 \subseteq N_2$ & $\beta_2|_{N_1} = \beta_1$

The set $\mathcal{S}$ of such pairs is non-empty $(\because (N, \beta) \in \mathcal{S})$ & it is closed under unions of chains

So by Zorn's lemma $\exists$ a maximal element (N', β') in $\mathcal{S}$

If $N' = M$ we're done.

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\beta"'] & M \arrow[dl, "\beta'"] \\ Q & \end{tikzcd}$$ ✓

So we may assume that $\exists m \in M \setminus N'$. Look at $N'' = Rm + N'$.

Let $I = \{r \in R;\ rm \in N'\}$, a left ideal.

Make a map $\theta: I \to Q$ $\quad \theta(r) = \beta'(\underbrace{rm}_{N'}) \in Q$

By hyp. can extend θ to $\delta: R \to Q$

Let $N'' = Rm + N'$ Define $\beta'': N'' \to Q$ $\beta''(rm + n') = \delta(r) + \beta'(n')$

Notice β'' is well-defined (check)

By construction, $\beta''|_{N'} = \beta'$ Contradicting maximality of (N', β'). ∎

Corollary: Let $R = \mathbb{Z}$. An R-module M is injective $\iff$ M is divisible

Proof: By A2, Injective $\Rightarrow$ divisible.

Suppose that M is divisible. Use Baer's criterion

Let $I \trianglelefteq \mathbb{Z}$ So $I = n\mathbb{Z}$ some $n \geq 0$

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "\beta"'] & \mathbb{Z} \arrow[dl, "\exists ? \beta'"] \\ M & \end{tikzcd}$$

If $I = (0)$ $\beta' = 0$ works

If $n \neq 0$ let $m \in M$ be $\beta(n)$

$\because M$ is divisible $\exists x \in M$ st $nx = m$. Define

$\beta': \mathbb{Z} \to M$, $1 \mapsto x$ $\quad \beta'(n) = nx = m = \beta(n)$ ✓ ∎

Cor: If M is an injective $\mathbb{Z}$-module & $K \leq M$ then M/K is injective.

Proof: $x+K \in M/K$, $x \in M$, $n \in \mathbb{Z}$, $n>0$ $\exists y \in M$ st $ny = x$
so $n(y+K) = x+K \Rightarrow M/K$ divisible.

"Enough Injectives"

An ~~_~~ abelian category $\mathcal{A}$ has enough projectives if $\forall A \in Ob(\mathcal{A})$ $\exists$ a projective obj. P & an epi $f: P \twoheadrightarrow A$. It has enough injectives if $\forall A \in Ob(\mathcal{A})$ $\exists$ an inj. obj. Q & a mono $A \xhookrightarrow{f} Q$
We'll see that R-Mod, R a ring, has enough injectives.

Case I: $R = \mathbb{Z}$

$\underset{\substack{\| \\ Ab}}{\mathbb{Z}\text{-Mod}}$ has enough injectives

Proof: Let A be an Abelian group. Then $\exists\ \mathbb{Z}^I \twoheadrightarrow A$
So $A \cong \mathbb{Z}^I/K$ $K \leq \mathbb{Z}^I$ Now $\mathbb{Z}^I \hookrightarrow \mathbb{Q}^I$

$K \leq \mathbb{Z}^I \leq \mathbb{Q}^I$

So $\underset{\substack{\text{is} \\ A}}{\mathbb{Z}^I/K} \leq \underset{\substack{\uparrow \\ \text{injective by cor}}}{\mathbb{Q}^I/K}$ So $A \hookrightarrow \mathbb{Q}^I/K$ which is injective

We'll lift this result to R-Mod. Set-up: S, R rings (ultimately, we'll take $S = \mathbb{Z}$)

- F is an (S,R)-bimodule i.e. F has structure as a left S-mod & as a right R-mod
& we'll assume that F is a flat right R-mod
i.e. If $0 \to M \to N$ is an exact sequence of left R-mod then

$0 \longrightarrow F \otimes_R M \longrightarrow F \otimes_R N$

is an exact seq of ab. grps

Aside NC tensor product
R ring T right R-Mod, L left R-mod
$T \otimes_R L$ ab grp $tr \otimes l = t \otimes rl$

Remark: If M is a left S-mod then we define $\tilde{M} := \mathrm{Hom}_S(F, M)$
$\tilde{M}$ is a left R-mod via $(\underset{R}{r} \cdot \underset{\mathrm{Hom}_S(F,M)}{\varphi})(\underset{F}{x}) = \varphi(x \cdot r)$

Injective Production Lemma: under this set-up if M is an inj. left S-mod Then $\tilde{M}$ is an inj. left R-mod.

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Proof: Must show $\mathrm{Hom}_R(-, \tilde{M})$ is exact.
In fact we know it is enough to show that if $0 \to A \xrightarrow{f} B$ is exact $A, B \in \mathrm{Ob}(R\text{-Mod})$ then $\mathrm{Hom}_R(B, \tilde{M}) \to \mathrm{Hom}_R(A, \tilde{M}) \to 0$ is exact.
$$\psi \longmapsto \psi \circ f$$

So let $0 \to A \xrightarrow{f} B$ be exact. Must show
$$\mathrm{Hom}_R(B, \mathrm{Hom}_S(F, M)) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(F, M)) \longrightarrow 0$$
$$\psi \longmapsto \psi \circ f$$
is exact.
$$\mathrm{Hom}_R(B, \mathrm{Hom}_S(F, M)) \cong \mathrm{Hom}_S(F \otimes_R B, M)$$
$$\psi : B \to \mathrm{Hom}_S(F, M)$$
$$b \longmapsto \psi(b) : F \to M$$
$$F \otimes_R B \longrightarrow M$$
$$\theta \otimes b \longmapsto \psi(b)(\theta)$$
Exercise: We have a map
$$\mathrm{Hom}_S(F \otimes_R B, M) \longrightarrow \mathrm{Hom}_S(F \otimes_R A, M)$$
$$\psi : F \otimes B \to M \to \hat{\psi}(\theta \otimes a) = \psi(\theta \otimes f(a))$$
& the isomorphisms give a commuting diagram

(*)
$$\begin{array}{ccc} \mathrm{Hom}_R(B, \mathrm{Hom}_S(F, M)) & \longrightarrow & \mathrm{Hom}_R(A, \mathrm{Hom}_S(F, M)) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}_S(F \otimes_R B, M) & \longrightarrow & \mathrm{Hom}_S(F \otimes_R A, M) \end{array}$$
So it suffices to show $\mathrm{Hom}_S(F \otimes_R B, M) \to \mathrm{Hom}_S(F \otimes_R A, M) \to 0$ is exact.
Since M is an injective(3) left module & F is flat(2) as a right R-mod
We know
1) $0 \to A \xrightarrow{f} B$ exact in R-mod
$\Downarrow$
2) $0 \to F \otimes_R A \xrightarrow{id \otimes f} F \otimes_R B$ is exact in S-Mod
$\Downarrow$
3) $\mathrm{Hom}_S(F \otimes_R B, M) \to \mathrm{Hom}_S(F \otimes_R A, M) \to 0$ is exact $\quad \psi \mapsto \hat{\psi}$
The result follows from (*) ie $\tilde{M}$ is an injective left R-mod. ∎

"Sometimes it's good to get your hands dirty.
What does it mean to get your hands dirty? Talk."

For us we'll take $S = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$ $\leftarrow$ injective S-module
F = free right R-module (free $\Rightarrow$ flat)
Define in this set-up if F is a right R-module $F^* = \operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$
is the Pontryagin dual F^* is a left R-mod

Remark: If A is a left ~~Z~~ or right R-mod **THEN**
$A \hookrightarrow A^{**}$ $\quad m \longmapsto e_m : A^* \to \mathbb{Q}/\mathbb{Z}$
$\qquad e_m(f) = f(m)$
Why is $m \mapsto e_m$ an injection?
Suppose $\exists m \in A \setminus \{0\}$ st $e_m = 0$. $\Rightarrow f(m) = 0$ $\forall f : A \to \mathbb{Q}/\mathbb{Z}$
Let $C = \mathbb{Z}m \subseteq A$
Claim: $\exists$ nat. trans. $g : C \to \mathbb{Q}/\mathbb{Z}$
Easy: possibilities $C \cong \mathbb{Z}$ ✓ $C \cong \mathbb{Z}/n\mathbb{Z}$ ✓

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d] & A \arrow[dl, dashed, "\tilde{g}"] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

$\mathbb{Q}/\mathbb{Z}$ $\leftarrow$ injective $\qquad$ So the claim follows

Cor: R ring, R-Mod has enough injectives

Proof: 1) If A is a right R-mod Then $\exists$ a free right R-mod F st $F \xrightarrow{\text{onto}} A$. Since $\mathbb{Q}/\mathbb{Z}$ is injective $\mathbb{Z}$-mod & A, F are $\mathbb{Z}$-mod, $0 \to \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is exact
flat injective
ie $A^* \hookrightarrow F^*$
By IPL this is an injective left R-mod
So, we see, 2) any ~~R~~ left R-mod of the form A^*, A a right R-module, embeds in an injective
3) Now let A be a left R-Module $A \hookrightarrow A^{**} \hookrightarrow F^*$ injective
$(A^*)^*$ embeds in an injective left R-mod by 2) $\Rightarrow A \hookrightarrow$ injective ▢

Nice fact: Any R-mod A has a unique minimal injective resolution

Def: Let R be a ring. $M \subseteq E$ be a left R-mod
We say that M is an <u>essential submodule</u> of E (or E is an essential extension of M) If $M \cap N \neq (0)$ $\forall$ $(0) \neq N \subseteq E$

unfortunately we're both in the past... or present?
oh we're running out of time too.

ProP: 1) Given a ring R $M \leq F$ modules $\exists$ a max^l sub. mod. $M \subseteq E \subseteq F$ with $M \subseteq E$ essential (A2)
2) If F is injective then E is injective (A2)
3) $\exists$ up to iso a unique essential ext. E of M that is an inj. R-mod. E : injective envelope of M write $E(M)$ for this

Idea after 3: $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(Q_1) \longrightarrow E(Q_2)$

$Q_1 = \operatorname{coker}(M \to E(M))$
$Q_2 = \operatorname{coker}(E(M) \to E(Q_1))$

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Proof: 1), 2) on A2
3) First, we know since R-Mod has enough injectives, $\exists F$ injective $M \overset{i}{\hookrightarrow} F$
WLOG $M \subseteq F$. By 1) & 2) $\exists E$ essential $M \subseteq E \subseteq F$, E inj. so we at least have existence
To see uniqueness, suppose we have $M \overset{\alpha_i}{\hookrightarrow} E_i$, E_1, E_2 inj. & essential ext. of M

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"] & E_1 \arrow[dl, "\beta"] \\ & E_2 & \end{tikzcd}$$

$\beta \circ \alpha_1 = \alpha_2$ so $\ker(\beta|_{\alpha_1(M)}) = (0)$ $\because \alpha_2$ is 1-1

Claim: $\ker(\beta) = (0)$, i.e. β is 1-1
Why? $\alpha_1(M) \subseteq E_1$ essential $\alpha_1(M) \cap \ker(\beta) = (0)$: since $\alpha_1(M) \subseteq E_1$ is ess.
$\Rightarrow \ker(\beta) = (0)$
So $E_1 \xrightarrow{\beta} E_2$ is 1-1 $\Rightarrow \beta(E_1) \subseteq E_2$ inj. $\Rightarrow \exists E_2'$ s.t. $\beta(E_1) \oplus E_2' = E_2$
Now $\alpha_2(M) = \beta \circ \alpha_1(M) \subseteq \beta(E_1)$ & $\alpha_2(M) \subseteq E_2$ is ess. So if $E_2' \neq (0)$ then $\alpha_2(M) \cap E_2' \neq (0)$ so $\beta(E_1) \cap E_2' \neq (0)$ $\times$ So $E_2' = (0)$
So $\beta(E_1) = E_2 \Rightarrow \beta$ is 1-1 & onto $\Rightarrow E_1 \cong E_2$. ∎

Remark: If $\{I_j\}_{j \in J}$ is a set of injective mod then $\prod_{j \in J} I_j$ is inj.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl] \arrow[ddl, dashed] \\ & \prod I_j \arrow[d] & \\ & I_j & \end{tikzcd}$$

Remark: A direct sum of inj. need not be injective

Thm [Bass]. Let R be a commutative ring. Then R is noetherian $\Leftrightarrow$ every direct sum of injectives is again injective.

Proof (Sketch): R not Noetherian $I_1 \subsetneq I_2 \subset I_3 \subset \cdots$

Let $E_n = E(R/I_n)$. Then $E := \bigoplus_{n=1}^{\infty} E_n$ is not inj.

Why? Let $I = \bigcup_{n=1}^{\infty} I_n \subseteq R$ Let $f_n : I \hookrightarrow R \longrightarrow R/I_n \hookrightarrow E(R/I_n)$

This gives us a map $f : I \longrightarrow \prod E(R/I_n)$ $x \longmapsto (f_1(x), f_2(x), \dots)$

key observation: f actually maps into $\bigoplus E(R/I_n)$

Why? $x \in I \Rightarrow x \in I_n$ $\forall n \gg 0 \Rightarrow f_n(x) = 0$ $\forall n \gg 0$

Now if E is injective, then $\exists \beta : R \to E$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[dl, "\beta"] \\ & E & \end{tikzcd}$$

$\exists m$ st $\beta(1) \in E_1 \oplus E_2 \cdots \oplus E_m \oplus (0) \oplus (0) \oplus \cdots$

$\Rightarrow \beta(r) = r\beta(1) \in E_1 \oplus \cdots \oplus E_m \oplus (0) \oplus \cdots$

Pick $x \in I_{m+1} \setminus I_m \Rightarrow f_{m+1}(x) \in E_{m+1} \neq (0)$ $\#$

<u>Other direction</u>: Exercise: M f.g. $\Rightarrow \mathrm{Hom}_R(M, \bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I} \mathrm{Hom}_R(M, N_i)$

Idea if R is noeth $J \subseteq R$ ideal then J f.g.

If the N_i are inj $\mathrm{Hom}(J, N_i) \longrightarrow \mathrm{Hom}(R, N_i)$ is onto $\forall i$

So

$$\begin{tikzcd} \mathrm{Hom}(J, \oplus N_i) \arrow[r, phantom, "\cong"] \arrow[d] & \oplus \mathrm{Hom}(J, N_i) \arrow[d] \\ \mathrm{Hom}(R, \oplus N_i) \arrow[r, "\cong"] & \oplus \mathrm{Hom}(R, N_i) \end{tikzcd}$$

Now Baer's crit gives $\oplus_i N_i$ is inj. ∎

Bass' thm is very useful when studying injectives over a Noetherian ring. An injective mod E is <u>decomposable</u> if $E = E' \oplus E''$, E', E'' nonzero; otherwise it is <u>indecomposable</u>.

For a comm noetherian ring R every inj. R-mod E is of the form $E \cong \bigoplus_{j \in J} E_j$, E_j inj. & indec.

Moreover $\exists$ a bij $\mathrm{Spec}(R) \longleftrightarrow$ isom. classes of indec. inj.

$$\mathfrak{p} \longmapsto E(R/\mathfrak{p})$$

Why? If E is indec & inj. Pick $x \in E$ with maximal annihilator $\mathrm{Ann}(x) = \{r \in R ; rx = 0\}$. Usual trick $\Rightarrow$ $\mathrm{Ann}(x) = \mathfrak{p}$ So

$$\begin{tikzcd} R/\mathfrak{p} \arrow[r, "\cong"] & Rx \arrow[r, hook] \arrow[d, hook] & E(R/\mathfrak{p}) \\ & E \arrow[ur, "\cong"'] & \end{tikzcd}$$