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Projective Modules

R ring, R-Mod left R-Modules, Mod(R) = right R-modules

Recall that an R-module P is <u>projective</u> if $\mathrm{Hom}(P,-): R\text{-Mod} \to R\text{-Mod}$ is exact. We know it is left exact. So it is equivalent to

$\exists \psi \quad g \circ \psi = \phi$

$$\begin{tikzcd} & P \arrow[d, "\phi"] & \\ M \arrow[ur, dashed, "\exists \psi"] \arrow[r, two heads, "g"] & N \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

Theorem: TFAE:

① ~~P is projective~~ We have diagram $(*)$

② Every short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ splits

③ $\exists Q$ st $P \oplus Q$ is free

④ The functor $\mathrm{Hom}(P,-)$ is exact

Proof:

①⇒②:

$$\begin{tikzcd} & P \arrow[dl, dashed, "s"'] \arrow[d, "id"] & \\ N \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

① ⇒ $\exists s$ st $g \circ s = id_P$

Now define $\Phi : P \oplus M \to N$

$(p, m) \longmapsto s(p) + f(m)$

Exercise: this is an isomorphism. So the s.e.s. splits.

①⇒③ Pick a free module F with $F \xrightarrow{g} P \longrightarrow 0$

Let $Q = \ker(F \xrightarrow{g} P)$. So $0 \to Q \to F \to P \to 0$

By (2), this splits, so $F \cong P \oplus Q$

③⇒④ Suppose that $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact

We know $\mathrm{Hom}(P,-)$ is left exact. We must show ~~TB~~

$$\mathrm{Hom}(P,M) \xrightarrow{\mathrm{Hom}(P,g)} \mathrm{Hom}(P,M'') \quad \text{is onto}$$

$$\psi \longmapsto g \circ \psi$$

ie we must show that if $h: P \to M'' \Rightarrow \exists h' : P \to M$ st $h = g \circ h'$.

Write $F = P \oplus Q$. Define $h_0 : F \to M''$ $h_0|_P = h$, $h_0|_Q = 0$.

Then because F is free, $\exists h_0' : F \to M$ st $g \circ h_0' = h_0$

$$\begin{tikzcd} & F \arrow[dl, "h_0'"'] \arrow[d, "h_0"] & \\ M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

Now let $h' = h_0'|_P$. Then $g \circ h' = g \circ h_0'|_P = h_0|_P = h$ So we have ④.
④ ⇒ ① is immediate

Ex $R = \mathbb{Z} \times \mathbb{Z}$ $P = \mathbb{Z} \times \{0\}$ P is not free
$(0,1) \cdot P = (0)$ so $\mathrm{Ann}(P) = \{0\} \times \mathbb{Z}$ so P is not free
But if $Q = \{0\} \times \mathbb{Z} \Rightarrow P \oplus Q = R$

Commutative situation
Let $(R, \mathcal{M})$ be a local ring ($\mathcal{M}$ unique maximal ideal) (R comm.)
Kaplansky: If P is projective R-module then P is free

Theorem: Let $(R, \mathcal{M})$ be ~~a~~ local & let P be f.g. projective. Then P is free.

Proof: Let $p_1, \dots, p_s$ be a minimal generating set for P (with s minimal)
Let $g: \underbrace{R \oplus \cdots \oplus R}_{s} \longrightarrow P \longrightarrow 0$

$$(0, \dots, 0, \underset{i\text{th}}{1}, 0, \dots, 0) \longmapsto p_i$$

Let $Q = \ker(g)$

$$0 \longrightarrow Q \xrightarrow{i} R^s \xrightarrow{g} P \longrightarrow 0$$

P projective $\Longrightarrow R^s \cong Q \oplus P$
Let $K = R/\mathcal{M}$ field $- \otimes K$ gives

$$R^s/\mathcal{M}R^s \cong P/\mathcal{M}P \oplus Q/\mathcal{M}Q$$

$$K^s = (R/\mathcal{M}R)^s \cong (P/\mathcal{M}P) \oplus (Q/\mathcal{M}Q)$$

Claim: $P/\mathcal{M}P \cong K^s$
Once we have the claim: $K^s = \underset{s \text{ dim'l}}{(P/\mathcal{M}P)} \oplus \underset{0 \text{ dim'l}}{Q/\mathcal{M}Q}$ $\Rightarrow Q = \mathcal{M}Q$

s dim'l $\Rightarrow$ 0 dim'l

$\mathcal{M} = J(R)$ so $Q = J(R)Q$ Nakayama $\Rightarrow Q = (0)$
(We need Q f.g. but this is OK because Q is a direct summand of a f.g. module)

Proof of claim: Suppose not. Then $P/MP \cong K^t$ for some $t<s$ $\because P/MP \cong K^s$
Pick $a_1, \ldots, a_t \in P$ so that $\bar{a}_1, \ldots, \bar{a}_t$ in P/MP form a $K = R/M$-basis
Now let $P_0 \subseteq P$. $P_0 = Ra_1 + \cdots + Ra_t \subsetneq P$

why? $t<s$ & s minimal among sizes of gen sets

Now let

$$N = P/P_0 \neq (0)$$

N is f.g. since P is f.g.

What is MN? $MN = (MP+P_0)/P_0 = P/P_0 = N$

Notice $MP+P_0 = P$ why? $\overline{P_0} = P/MP$

Nakayama $\Rightarrow$ $N = (0)$ ※ ■

Remark. If R is a PID & P projective $\Rightarrow$ P is free

f.g. case: $P \cong R^m \oplus T$, T torsion

structure thm for f.g. modules over a P.I.D.

$$T = \oplus R/I, \ I \neq (0)$$

But we saw $\exists\, Q$ f.g. $P \oplus Q \cong R^L$

(Pick $R^L \xrightarrow{g} P \to 0$, let $Q = \ker(g)$ $L = \#$ of gens of P

Since Q is f.g. $Q \cong R^n \oplus T'$, T' torsion

So $P \oplus Q \cong (R^m \oplus T) \oplus (R^n \oplus T') = (R^m \oplus R^n) \oplus (T \oplus T') \cong R^L$

a free module cannot contain non zero torsion

$\Rightarrow T = T' = (0) \Rightarrow P$ free

(*) Theorem. Let R be a commutative noetherian ring. Let P be f.g. TFAE:
1) P is projective (module)
2) $P_\mathfrak{p} = P \otimes_R R_\mathfrak{p}$ is free $\forall\, \mathfrak{p} \in \operatorname{Spec}(R)$
3) $P_M = P \otimes_R R_M$ is free $\forall$ maximal ideals M

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Theorem [Bass]: Let R be a noetherian commutative ring & suppose that 0 & 1 are the only idempotents. Then if P is a projective R-module & P is not f.g. then P is free.

Recall: $\operatorname{Spec}(R) = \{\mathfrak{p} ; \mathfrak{p} \text{ a prime ideal}\}$ top. space

Zariski topology: closed sets: $\{\mathfrak{p} ; \mathfrak{p} \supseteq I\}$, $I \trianglelefteq R$

principal open sets $V(f) = \{\mathfrak{p} ; f \notin \mathfrak{p}\}$

Localization:

S mult. closed subset of R, $0 \notin S$

$S^{-1}R = \{s^{-1}r ; s \in S, r \in R\}$, $s^{-1}r = (r,s)$

$(r_1,s_1) \sim (r_2,s_2) \iff s_3(r_1 s_2 - s_1 r_2) = 0$ for some $s_3 \in S$

M is an R-module

$$S^{-1}M = M \otimes_R S^{-1}R$$

$\{s^{-1}m ; s \in S, m \in M\}$

(s,m) $\qquad (s_1,m_1) \sim (s_2,m_2) \iff s_3(s_1 m_2 - s_2 m_1) = 0$ for some $s_3 \in S$

$M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$, $R_{\mathfrak{p}} = S^{-1}R$, $S = \{x \in R ; x \notin \mathfrak{P}\}$

If $f \in R$ not nilpotent $M_f = R_f \otimes_R M$, $R_f = S^{-1}R$, $S = \{1, f, f^2, \dots\}$

Proof (of (*)):

(1) $\Rightarrow$ (2): If P is f.g. projective $\exists n \geq 1$, $R^n \xrightarrow[\text{epi}]{g} P$ If $Q = \ker(g)$

$0 \to Q \to R^n \to P \to 0 \Rightarrow R^n \cong Q \oplus P$

Now applying $- \otimes_R R_{\mathfrak{p}}$ we see

$$R^n \otimes_R R_{\mathfrak{p}} \cong (P \oplus Q) \otimes_R R_{\mathfrak{p}}$$

$$R^n_{\mathfrak{p}} = (R \otimes_R R_{\mathfrak{p}})^n \cong P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}}$$

So $P_{\mathfrak{p}}$ is a direct summand of a free $\Rightarrow P_{\mathfrak{p}}$ is projective $\Rightarrow P_{\mathfrak{p}}$ is free

($\because R_{\mathfrak{p}}$ is a local ring & $P_{\mathfrak{p}}$ is f.g.)

(2) $\Rightarrow$ (3) Clear ($\because \mathfrak{M}$ maximal $\Rightarrow \mathfrak{M}$ prime)

(3) $\Rightarrow$ (1) Assume $P_{\mathfrak{M}}$ is free (of finite rank) $\forall \mathfrak{M}$, maximal ideal of R

Recall: P is projective $\iff$ whenever $M \xrightarrow[\text{epi}]{g} M' \to 0$

$\Rightarrow \quad \mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(P,M') \longrightarrow 0$ is exact

$\psi \longmapsto g \circ \psi$

(i.e. $\mathrm{Hom}(P, -)$ is exact)

Strategy: Let $M \xrightarrow[\text{epi}]{g} M'$ & we'll show $\mathrm{Hom}(P,M) \xrightarrow{\text{epi}} \mathrm{Hom}(P,M')$

$M \xrightarrow{g} M' \to 0$ Let $\mathfrak{M}$ be a maximal ideal $\because \otimes_R R_{\mathfrak{M}}$ is right exact

$\Rightarrow M_{\mathfrak{M}} = M \otimes_R R_{\mathfrak{M}} \xrightarrow{g \otimes id} M'_{\mathfrak{M}} = M' \otimes_R M_{\mathfrak{M}} \longrightarrow 0$

$\because P_{\mathfrak{M}}$ is projective

$\Rightarrow \mathrm{Hom}_{R_{\mathfrak{M}}}(P_{\mathfrak{M}}, M_{\mathfrak{M}}) \xrightarrow{g \circ -} \mathrm{Hom}_{R_{\mathfrak{M}}}(P_{\mathfrak{M}}, M'_{\mathfrak{M}})$ is onto

Assignment 3: $\mathrm{Hom}_{R_{\mathfrak{M}}}(P_{\mathfrak{M}}, M_{\mathfrak{M}}) \cong \mathrm{Hom}_R(P,M) \otimes_R R_{\mathfrak{M}} = \mathrm{Hom}_R(P,M)_{\mathfrak{M}}$

P f.g. $\underset{\exists m,n \geq 1}{\iff}$ $R^m \to R^n \to P \to 0$

So from A3 $\mathrm{Hom}(P,M)_{\mathfrak{m}} \xrightarrow{g\circ} \mathrm{Hom}(P,M')_{\mathfrak{m}}$ is onto $\forall$ maximal ideal $\mathfrak{m}$ — $(*)$

Claim: R comm, noeth

Let M_1 & M_2 be f.g. modules & suppose $M_1 \xrightarrow{g} M_2$ is a homomorphism

If $(M_1)_{\mathfrak{m}} \xrightarrow{g} (M_2)_{\mathfrak{m}}$ is onto $\forall \mathfrak{m}$ maximal ideal $\Rightarrow g$ is onto

Once we have this claim, Then $(*) \Rightarrow \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,M')$ is onto when M, M' f.g.

Proof of claim: Let $K = \mathrm{coker}(g)$ so $M_1 \xrightarrow{g} M_2 \to K \to 0$ is exact

So $-\otimes_R R_{\mathfrak{m}}$ is (right) exact $(M_1)_{\mathfrak{m}} \xrightarrow{g} (M_2)_{\mathfrak{m}} \to K_{\mathfrak{m}} \to 0$ is exact $\forall$ maximal $\mathfrak{m}$

Since $(M_1)_{\mathfrak{m}} \xrightarrow[\text{epi}]{g} (M_2)_{\mathfrak{m}} \to 0$

$\Rightarrow K_{\mathfrak{m}} = (0)$ $\forall \mathfrak{m}$ maximal $\Rightarrow$ for each $k \in K$ $1^{-1}k \sim 1^{-1}0$ in $K_{\mathfrak{m}}$

$\Leftrightarrow \exists s \notin \mathfrak{m}$ st $sk = 0$

Since M_2 ~~is~~ is f.g $\Rightarrow K \cong M_2/\mathrm{Im}(g)$ is f.g

Let $k_1, \dots, k_r$ be a set of gens. If $\mathfrak{m}$ is maximal $\exists s_1, \dots, s_r \notin \mathfrak{m}$ st $s_i k_i = 0$. Let $s = s_1 \cdots s_r \notin \mathfrak{m} \Rightarrow s k_i = 0$ $\forall i \Rightarrow sk = 0$ $\because k_1, \dots, k_r$ generate

Conclusion: $\forall \mathfrak{m}$ maximal ideal of R $\exists s_{\mathfrak{m}} \notin \mathfrak{m}$ st $s_{\mathfrak{m}} \cdot K = 0$.

Let $I = \{s \in R;\ s \cdot K = 0\} = \mathrm{Ann}(K)$ an ideal.

If $I = R \Rightarrow 1 \cdot K = (0) \Rightarrow K = (0) \Rightarrow g$ is onto, as desired.

So WLOG I is proper $\Rightarrow \exists \mathfrak{m}$ maximal $I \subseteq \mathfrak{m}$

But $s_{\mathfrak{m}} \notin \mathfrak{m}$ is in I ⫽

Conclude that If $\mathrm{Hom}(P,M)$ & $\mathrm{Hom}(P,M')$ are f.g. & $M \xrightarrow{g} M' \to 0 \Rightarrow \mathrm{Hom}(P,M) \xrightarrow{g\circ -} \mathrm{Hom}(P,M') \to 0$

Notice if $P = R^n$, $M = \langle m_1, \dots, m_s \rangle$ $\Phi_{r,i} : R^n \to M$

$e_i \mapsto m_{r(i)}$ $r(i) \in \{1, \dots, s\}$

$e_j \mapsto 0$ $j \neq i$

$\Phi(e_1) = a_{11}m_1 + \cdots + a_{1s}m_s$

$\vdots$

$\Phi(e_n) = a_{n1}m_1 + \cdots + a_{ns}m_s$

$\Phi = a_{11}\Phi_{1,1} + \cdots + a_{ns}\Phi_{sn}$ (ns many)

Because P is locally free (f.g.) & M, M' are f.g. you can show $\mathrm{Hom}(P,M)$ & $\mathrm{Hom}(P,M')$ are f.g. (Exercise)

Conclusion: M, M' f.g

$\Rightarrow \mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(P,M')$ onto

Now take $M = R^n$, $M' = P$ $\Rightarrow$

$$\begin{tikzcd} R^n \arrow[r, "g"] & P \arrow[r] & 0 \\ & P \arrow[u, "id"'] \arrow[ul, dashed, "\exists s"] & \end{tikzcd}$$

$\Rightarrow P \oplus \ker(g) \cong R^n \longrightarrow P$ projective.

From here $\mathfrak{p} \rightsquigarrow P_{\mathfrak{p}}$ free

$\cong R_{\mathfrak{p}}^d \quad d \geq 1$

$$\operatorname{Spec}(R) \longrightarrow \mathbb{Z}$$
$$\mathfrak{p} \longmapsto d(\mathfrak{p}) = \operatorname{rank}(P_{\mathfrak{p}})$$

A3: show this map is c/s

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R comm. noeth. ring

P f.g. proj ~~mod~~ R-mod

We saw that in this case P proj., f.g.

$\Updownarrow$

$P_{\mathfrak{p}} = P \otimes_R R_{\mathfrak{p}}$ is free of finite rank $\forall \mathfrak{p} \in \operatorname{Spec} R$

$\Updownarrow$

$P_{\mathfrak{m}} = P \otimes_R R_{\mathfrak{m}}$ free $R_{\mathfrak{m}}$-mod of fin. rank $\forall$ max ideal $\mathfrak{m}$

Remark: If P is f.g., R comm noeth then if $P_{\mathfrak{m}}$ free $\Rightarrow \exists f \in R \setminus \mathfrak{m}$ st P_f is free as an R_f-mod

Why? Since P is f.g. as an R-mod, $P = \langle p_1, \dots, p_m \rangle = Rp_1 + \dots + Rp_m$

By assumption $P_{\mathfrak{m}}$ is free

$P_{\mathfrak{m}} = \{ s^{-1}p \,;\, s \notin \mathfrak{m}, p \in P \}$

Pick $s_1^{-1}q_1, \dots, s_d^{-1}q_d \in P_{\mathfrak{m}}$ st $P_{\mathfrak{m}} = \bigoplus_{i=1}^{d} R_{\mathfrak{m}} s_i^{-1} q_i$

Then $q_1, \dots, q_d \in P$ form a basis for $P_{\mathfrak{m}}$ ie $P_{\mathfrak{m}} = \oplus_{i=1}^d R_{\mathfrak{m}} q_i$

By assumption for $i = 1, \dots, m$ $1^{-1}p_i = p_i \in P_{\mathfrak{m}}$ So

$$p_i = (\mu_{i1}^{-1} r_{i1}) q_1 + \dots + (\mu_{id}^{-1} r_{id}) q_d \quad \mu_{ij} \in R \setminus \mathfrak{m} \quad r_{ij} \in R$$

Pick $s \in R \setminus \mathfrak{m}$ st $s\mu_{ij}^{-1} \in R \ \forall i,j$

Then $sp_i \in Rq_1 + \dots + Rq_d \ \forall i \Rightarrow \underset{P}{\overset{\wedge}{p_i}} \in \underbrace{R_s q_1 + \dots + R_s q_d}_{Q_s}$

So let $Q = Rq_1 + \dots + Rq_d \subseteq P \Rightarrow Q_s = P_s$

Now consider $R_s^d \longrightarrow Q_s = P_s \quad e_i \longmapsto q_i$

Let $K = \ker(R_s^d \to P_s)$ Then $0 \to K \to R_s^d \to P_s \to 0$

Notice when we localize to $R_{\mathfrak{m}} \Rightarrow 0 \to K_{\mathfrak{m}} \to R_{\mathfrak{m}}^d \to P_{\mathfrak{m}} \to 0$

Here we know $R_{\mathfrak{m}}^d \to P_{\mathfrak{m}}$ is an isom $\Rightarrow K_{\mathfrak{m}} = (0)$

$\because R$ is Noeth $\Rightarrow R_s$ is noeth $\Rightarrow K$ is f.g. as an R_s-mod.

Now Exercise: $K_{\mathfrak{m}} = (0)$

$\Rightarrow \exists s' \notin \mathfrak{M}$ st $K_{s'} = (0)$

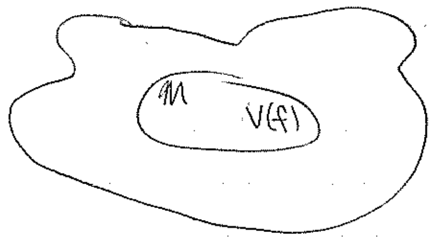
Idea $K_{\mathfrak{m}} = \{ s^{-1}k \,,\, s \in R \setminus \mathfrak{M},\ k \in K \}$

$K_{\mathfrak{m}} = (0) \Rightarrow \forall k \in K\ \exists s \in R \setminus \mathfrak{M}$ st $sk = 0$

Now if we invert ss' we get $0 \to \underset{(0)}{\overset{}{K_{s'}}} \to R^d_{ss'} \to P_{ss'} \to 0$

So $P_{ss'} = R^d_{ss'}$. Taking $f = ss'$ we see $P_f \cong R^d_f$

Spec(R)



$\mathfrak{M}$ max ideal $\longrightarrow f \notin \mathfrak{M}$ st $P_f \cong R^d_f$

$\mathrm{Spec}(R_f) = \{ \mathfrak{P} ; f \in \mathfrak{P} \} - V(f) \subseteq \mathrm{Spec}(R)$ open

Notice for every $\mathfrak{P} \in V(f)$ $R_\mathfrak{P}$ is a localization of R_f

so $P_f \cong R^d_f \to P_\mathfrak{P} \cong R^d_\mathfrak{P}$

$P_f \otimes_{R_f} R_\mathfrak{P} \cong (R^d_f \otimes_{R_f} R_\mathfrak{P}) \cong R^d_\mathfrak{P}$

What is this saying?

P f.g. proj., R comm noeth

Recall free mod over a comm ring have a well-defined rank

$$\psi : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}$$
$$\mathfrak{P} \longmapsto \mathrm{rank}_{R_\mathfrak{P}}(P_\mathfrak{P})$$
$$V(f) \ni \mathfrak{P} \longmapsto \overset{d}{\text{constant}}$$

Ex ψ is cts

Cor: If Spec(R) is connected $\Rightarrow \psi$ is constant

In this case we can define rank(P) to be the image of ψ

Ex Spec(R) is disconnected

$\Updownarrow$

$R \cong R_1 \times R_2$, R_1, R_2 nonzero

$\Updownarrow$

R has an idempotent e, $e^2 = e$ $e \neq 0, 1$

Since rank is additive for free mod, if Spec(R) is conn. then

rank $(P \oplus Q)$ = rank (P) + rank(Q)

not all proj. are free

A f.g. proj. mod

P is called <u>stably free</u> if $\exists m, n \geq 1$ st $P \oplus R^m \cong R^n$

Equivalently, $0 \to R^m \to R^n \to P \to 0$

Swan's example

Let $A = \mathbb{R}[x,y,z]/(1-x^2-y^2-z^2)$

We have a surjection $g: A^3 \twoheadrightarrow A$ $(a,b,c) \longmapsto ax+by+cz$

Let $P = \ker(g)$ $\qquad (rx, ry, rz) \longmapsto rx^2+ry^2+rz^2 = r$

So $0 \to P \to A^3 \xrightarrow{g} A \to 0$

$(x,y,z) \xleftarrow{s} 1$

So $A^3 \cong P \oplus A \Rightarrow P$ is stably free

Thm (Swan) P is not free!

Proof: If P were free $P \cong A^2$ & $P \subseteq A^3$

So $P = \langle (f_1, f_2, f_3), (g_1, g_2, g_3) \rangle \subseteq A^3$

Now $A^3 = P \oplus \underbrace{s(A)}_{\langle (x,y,z) \rangle}$

So $A^3 = \langle (f_1, f_2, f_3), (g_1, \ldots), (x,y,z) \rangle$

So $(1,0,0) = a_1(f_1, f_2, f_3) + b_1(g_1 \ldots) + c_1(\qquad)$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ x & y & z \end{pmatrix}$$

If we plug in any pt $(\alpha, \beta, \gamma) \in S^2$.

So $\det \begin{pmatrix} f_1(\alpha,\beta,\gamma) & & \\ & & \\ \alpha & \beta & \gamma \end{pmatrix} \neq 0 \quad \forall (\alpha, \beta, \gamma)$

Now ~~view~~ view $(f_1, f_2, f_3): S^2 \to \mathbb{R}^3$ cts

Claim: any cts map $\psi: S^2 \to \mathbb{R}^3$ has the property that $\exists p \in S^2$ $\lambda \in \mathbb{R}$ st $\psi(p) = \lambda p$ $\times$ take $p = (\alpha, \beta, \gamma)$.

"proof of claim" Reduction 1 WLOG $\psi: S^2 \to \mathbb{R}^3 \setminus \{0\}$

Reduction 2 Replace $\psi(p)$ by $\psi(p)/\|\psi(p)\| : S^2 \to S^2$

$\vdots$

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Vector bundles

Let S be a connected compact real manifold.
A (real) vector bundle over S of rank n is a top. space V with a continuous map $\pi: V \longrightarrow S$ st
1) $\forall x \in S$ $\pi^{-1}(x)$ is a real v.s. of dim n
2) $\forall x \in S$ $\exists$ an open nbd U of $x \in S$ & a homeomorphism $\varphi: U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ st $\pi \circ \varphi = p$ where $p: U \times \mathbb{R}^n \longrightarrow U$ is projection
3) And $\forall y \in U$, $\varphi|_{\{y\}\times\mathbb{R}^n} \; \{y\}\times\mathbb{R}^n \longrightarrow \pi^{-1}(\{y\})$ is a lin. iso. of v.s.

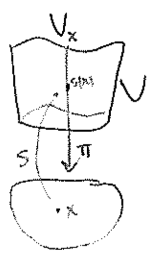
A vector bundle is trivial if $V \cong S \times \mathbb{R}^n$

vector bundles $\longleftrightarrow$ projective modules
S compact connected $\mathbb{R}$-man. $\longleftrightarrow$ $C(S)$ = cts functions on S, $f: S \xrightarrow{\text{cts}} \mathbb{R}$

Given a vector bundle $V \xrightarrow{\pi} S$ over S of rank n
Define a $C(S)$-module $P(V)$ as follows

Def) Let $\pi: V \to S$ be as before. A section of π is a cts map $s: S \to V$ st $\pi \circ s = id_S$. $P(V) = \{s: S \longrightarrow V \text{ section}\}$

Remark: $P(V)$ is a $C(S)$-module
Why?



If $s,t \in P(V)$, $(s+t)(v) = s(v)+t(v)$
So $P(V)$ is a $C(S)$-module.

Thm (Swan): If V is a vector bundle of rank n then $P(V)$ is a proj. $C(S)$-module of rank n. Moreover
$$V \leftrightarrow P(V)$$
gives an equivalence of categories
vector bundles over S $\longleftrightarrow$ f.g. proj. $C(S)$-mod
trivial v.b. $\longleftrightarrow$ free mod