2016 02 04

Abelian Categories

A pre-additive category $\mathcal{C}$ is just a category in which $\forall A,B \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(A,B)$ has the structure of an abelian group

Remark: $\exists 0_{A,B} : A \to B$, $0_{A,B} + f = f \quad \forall f : A \to B$ (non-empty)

If A, B, C objects

$$\begin{array}{ccccc} \circ_{A,B,C} : Hom_{\mathcal{C}}(B,C) & \times & Hom_{\mathcal{C}}(A,B) & \longrightarrow & Hom_{\mathcal{C}}(A,C) \\ \rotatebox{90}{$\in$}\; g & \circ_{A,B,C} & \rotatebox{90}{$\in$}\; f & \longmapsto & g \circ f \end{array}$$

This map is bilinear, i.e. $\circ_{A,B,C} : Hom_{\mathcal{C}}(B,C) \otimes_{\mathbb{Z}} Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{C}}(A,B)$ is a homomorphism

Ex Ring R Category 1-object R Morphisms φ_r, $r \in R$, $\varphi_r : R \to R$
$\varphi_r(x) = rx$
$\varphi_0 = 0_{R,R}$, $\varphi_r + \varphi_s = \varphi_{r+s}$, $(\varphi_r + \varphi_s) \circ \varphi_t$
$\varphi_{rt} + \varphi_{st} = \varphi_r \circ \varphi_t + \varphi_s \circ \varphi_t$

So this category is preadditive.

Remark: If $\mathcal{C}, \mathcal{D}$ are pre-additive categories,
$A,B \in Ob(\mathcal{C})$ $F : \mathcal{C} \to \mathcal{D}$ is called additive if the map $f \mapsto F(f)$ gives a homomorphism from $Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(FA, FB)$.

Def) A pre-additive category is additive if all finite (including empty) products & coproducts exist

Remark: $\mathcal{C}$ additive, $A,B \in Ob(\mathcal{C})$

$$\underset{A \times B}{\overset{\shortparallel}{A \prod B}} \cong \underset{A \oplus B}{\overset{\shortparallel}{A \amalg B}}$$

Proof:

$$\begin{tikzcd} & A \prod B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] \arrow[dd, "\theta"] & \\ A \arrow[dr, "i_A"'] & & B \arrow[dl, "i_B"] \\ & A \amalg B & \end{tikzcd}$$

θ part doesn't commute

$\theta \cong$ should be $i_A \circ p_A + i_B \circ p_B$

$$\begin{tikzcd} & A \arrow[ddl, bend right, "id_A"'] \arrow[d, "\exists \mu_A"] \arrow[ddr, bend left, "0_{A,B}"] & \\ & A \Pi B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A & & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[ddl, bend right, "0_{A,B}"'] \arrow[d, "\exists \mu_B"] \arrow[ddr, bend left, "id_B"] & \\ & A \Pi B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A & & B \end{tikzcd}$$

$id_A = p_A \circ \mu_A \quad 0 = p_A \circ \mu_B$

$0 = p_B \circ \mu_A \quad id_B = p_B \circ \mu_B$

Claim $A \Pi B$ is coproduct

$$(*) \quad \begin{tikzcd} A \arrow[r, "\mu_A"] \arrow[dr, bend right, "f"'] & A \Pi B \arrow[d, dashed, "\exists! \theta?"] & B \arrow[l, "\mu_B"'] \arrow[dl, bend left, "g"] \\ & C & \end{tikzcd}$$

What should θ be?

θ should be $f \circ p_A + g \circ p_B$

Let's check that this θ works!

Must show $f = \theta \circ \mu_A$, $g = \theta \circ \mu_B$

$\theta \circ \mu_A = (f \circ p_A + g \circ p_B) \circ \mu_A$

$= f \circ p_A \circ \mu_A + g \circ p_B \circ \mu_A = f \circ id_A + g \circ 0 = f$ ✓ similarly for

Let's show θ is unique.

Suppose θ & θ' both make diagram $(*)$ commute

$\theta \circ \mu_A = f \quad \theta' \circ \mu_A = f$

$\theta \circ \mu_B = g \quad \theta' \circ \mu_B = g$

Let $\psi : A \Pi B \to C$, $\psi = \theta - \theta'$. $\psi \circ \mu_A = 0$, $\psi \circ \mu_B = 0$

$$\begin{tikzcd} & A \Pi B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A \arrow[r, "\mu_A"] \arrow[dr, bend right, "0"'] & A \Pi B \arrow[d, "\psi"] & B \arrow[l, "\mu_B"'] \arrow[dl, bend left, "0"] \\ & C & \end{tikzcd}$$

Claim: $\mu_A \circ p_A + \mu_B \circ p_B = id_{A \Pi B}$

Once we have the claim,

$\psi = \psi \circ id_{A \Pi B} = \psi \circ (\mu_A \circ p_A + \mu_B \circ p_B)$

$= \underbrace{\psi \circ \mu_A}_{0} \circ p_A + \underbrace{\psi \circ \mu_B}_{0} \circ p_B = 0$

Why?

$$\begin{tikzcd} & A \Pi B \arrow[ddl, bend right, "p_A"'] \arrow[d, "\mu_A \circ p_A + \mu_B \circ p_B"] \arrow[ddr, bend left, "p_B"] & \\ & A \Pi B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A & & B \end{tikzcd}$$

$p_A \circ (\quad) = p_A$

But $id_{A \Pi B}$ also works in place of $\mu_A \circ p_A + \mu_B \circ p_B$, so uniqueness gives the claim. ∎

Remark: We also have a zero object

Why? Empty coproduct . . . initial object I — Empty product gives a final object

$$\begin{array}{c} I \\ \downarrow \exists! \\ N \end{array} \qquad \begin{array}{c} N \\ \downarrow \exists! \\ T \end{array}$$

Claim $I \cong T$. Henceforth we'll write 0 for the zero object.

$$\exists! \left(\begin{array}{c} T \\ \downarrow \\ I \\ \downarrow \exists! \\ T \end{array} \right.$$

Notice if $f: A \to B$ $\quad A \overset{f}{\underset{0_{AB}}{\rightrightarrows}} B \quad$ limit = Equalizer

Intuitively "$\{x \in A ; f(x) = 0\}$"

If the equalizer $A \overset{f}{\underset{0}{\rightrightarrows}} B$ exists, we call it the kernel of f.

Similarly, $\mathrm{coker}(f) := \varinjlim (A \rightrightarrows B)$ if it exists.

An additive category in which kernels & cokernels exist is called <u>pre-abelian</u>.

Abelian Categories

$A \hookrightarrow B$

<u>Def</u> A map $f: A \to B$ is called <u>monomorphism</u> "$\cancel{A}$" if $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2$.

$f: A \to B$ is an epimorphism if $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$

monomorphism + epimorphism $\not\Rightarrow$ isomorphism

Eg. $\mathcal{R}$ - category of commutative rings $\quad \mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \overset{h_1}{\underset{h_2}{\rightrightarrows}} R \quad$ i is a monomorphism

Claim $h_1 \circ i = h_2 \circ i \Rightarrow h_1 = h_2$

$h_1(n) = h_2(n) \quad \forall n \in \mathbb{Z}$

$h_1(b) h_1(\frac{1}{b}) = h_1(1) = 1 \quad \forall b \in \mathbb{Z} \setminus \{0\}$

$h_2(b) h_2(\frac{1}{b}) = 1$

$h_1(a/b) = h_1(a) h_1(1/b) = h_2(a) h_2(1/b) = h_2(a/b) \Rightarrow h_1 = h_2$

An <u>abelian category</u> is one in which every monomorphism & epimorphism is <u>normal</u>.

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$A \xrightarrow{f} B$ monomorphism

f is <u>Normal</u> if f is a kernel

Similarly $B \xrightarrow{g} C$ epimorphism is normal if g is a cokernel

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] \arrow[dr, "0"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Exercise: This implies

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \\ & \mathrm{Im}(f) \arrow[ur, "v"'] & \end{tikzcd}$$

u is an epimorphism
v is a monomorphism

$$A \xrightarrow{f} B \longrightarrow \underset{\mathrm{Coker}(f)}{\underset{\shortparallel}{B/\mathrm{Im}(f)}}$$

$\mathrm{Im}(f) = \ker(\mathrm{Coker}(f))$

<u>Example</u> R ring with 1, associative, not necessarily commutative

R-Mod = Category of left R-modules

is an abelian ~~group~~ category

Remark: In R-Mod

$$\ker(f) \overset{i}{\underset{0}{\rightrightarrows}} M \xrightarrow{f} N$$ is a monomorphism $\Leftrightarrow$ f is 1-1

$f \circ i = f \circ 0$ f monomorphism $\Rightarrow i = 0 \Rightarrow \ker(f) = (0)$

Similarly, epimorphisms are onto

<u>Diversion</u>

Mitchell's embedding lemma: Let $\mathcal{A}$ be a small abelian category. Then

$\exists F: \mathcal{A} \to R\text{-Mod}$, R ring F is full, faithful, exact

$A, B \in ob(\mathcal{A})$

$F: \mathrm{Hom}_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Hom}_R(FA, FB)$

- full: F is onto $\forall A, B$
- faithful: F is 1-1 $\forall A, B$

<u>exact</u>: If F is additive & if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact in $\mathcal{A}$

$\Rightarrow 0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact in R-Mod

$$\begin{tikzcd} M \arrow[r] \arrow[d, "v"'] \arrow[dr] & P \arrow[d] \\ S \arrow[r] & Q \end{tikzcd}$$

"Let's try to outthink this theorem.
I know you can't outthink a theorem"

What is exactness?

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Then what does it mean to say this is exact at B?

1) $g \circ f = 0$ &

2) the canonical map $\tilde{f} : \mathrm{Im}(f) \to \ker(g)$ is an isomorphism

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"'] & \circlearrowleft & B \\ & \mathrm{Im}(f) \arrow[ur, "v"'] & \end{tikzcd}$$

$\mathrm{Im}(f) = \ker(\mathrm{Coker}(f))$

$$\begin{tikzcd} & \mathrm{Im}(f) = \ker(\pi) \arrow[d, "i"] \\ A \arrow[ur, dashed, "\exists! \theta"] \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \arrow[d, shift right, "0"'] \arrow[d, shift left, "\pi"] \\ & \mathrm{Coker}(f) \end{tikzcd}$$

by universal property of kernel

$$\begin{tikzcd} & \mathrm{Im}(f) \arrow[dr, "i"] & & \\ A \arrow[ur, "\theta"] \arrow[rr, "f"] \arrow[rrr, bend right, "0"'] & & B \arrow[r, "g"] & C \end{tikzcd}$$

θ epimorphism
i monomorphism

$$0 = g \cdot f \Rightarrow g \circ i \circ \theta = 0 \Rightarrow g \circ i \circ \theta = 0 \circ \theta \Rightarrow g \circ i = 0 \because \theta \text{ is epi}$$

$$\begin{tikzcd} \mathrm{Im}(f) \arrow[r, dotted, "\exists \tilde{f}"] \arrow[d, "i"] & \ker(g) \arrow[dl] \\ B \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & C \end{tikzcd}$$

Next Remark $0 \xrightarrow{0} A \xrightarrow{f} B$ is exact at $A \iff f$ is a monomorphism

$B \xrightarrow{g} C \xrightarrow{0} 0$ is exact at $C \iff g$ is an epimorphism

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

<u>Mitchell's Theorem</u>

$\mathcal{A}$ small abelian category
$\int$
R-Mod

If we start with R-Mod, can we "recover" R.

Remark: If $\mathcal{A}$ is an abelian category $A \in \mathrm{Ob}(\mathcal{A})$
$\mathrm{Hom}_{\mathcal{A}}(A, A) = \mathrm{End}_{\mathcal{A}}(A)$ is a ring under $\circ$

In R-Mod R as a left R-Module

$\mathrm{End}_R(R) \cong R^{op}$ $\quad s * r = r \cdot s$

What's the homomorphism $\psi: R \longrightarrow R$

ψ is determined by $\psi(1)$. If $\psi(1) = s$, $\psi(r) = r \cdot \psi(1) = r \cdot s$

So $\psi = \Phi_s$ for some $s \in R$ where $\Phi_s(x) = xs$.

$\mathrm{End}_R(R) \cong \{\psi_s ; s \in R\} \cong R^{op}$

$\psi_s \circ \psi_r(x) = \psi_s(xr) = xrs = \psi_{rs}(x) \qquad \psi_s \circ \psi_r = \psi_{rs}$

$\exists R \not\cong S \quad R\text{-Mod} \cong S\text{-Mod}$

Ex $\quad R\text{-Mod} \cong M_n(R)\text{-Mod}$

$R^n \rightsquigarrow \mathrm{End}_R(R^n) \cong M_n(R^{op}) \quad$ So $\mathrm{End}_R(R^n)^{op} \cong M_n(R)$

Fix Notion ~~of~~ close to freeness

Let $\mathcal{A}$ be an abelian category. $M \in \mathrm{Ob}(\mathcal{A})$

$\mathrm{Hom}(M, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$

$B \longmapsto \mathrm{Hom}_{\mathcal{A}}(M, B)$

We say that M is a projective object of $\mathcal{A}$ if the functor $\mathrm{Hom}(M,-)$ is exact

What are projectives in R-Mod? $\mathrm{Hom}(P,-)$ is left exact check $\forall P$

When is $\mathrm{Hom}(P,-)$ right exact?

$M \xrightarrow{g} N \to 0$ exact $\Rightarrow \mathrm{Hom}(P, M) \xrightarrow{\psi \mapsto g \circ \psi} \mathrm{Hom}(P, N) \longrightarrow 0$ is exact

$$\begin{tikzcd} M \arrow[r, "g"] & N \arrow[r] & 0 \\ & P \arrow[ul, dashed, "\exists \psi"] \arrow[u, "\varphi"'] & \end{tikzcd}$$

Remark: P projective $\Rightarrow \exists Q$ st $P \oplus Q \cong R^I$

Idea!

$$\begin{tikzcd} R^I \arrow[r, two heads] & P \arrow[r] & 0 \\ & P \arrow[ul, dashed, "s"] \arrow[u, "id"'] & \end{tikzcd}$$

$0 \to Q \to R^I \underset{s}{\overset{\pi}{\rightleftarrows}} P \to 0 \Rightarrow R^I \cong Q \oplus P$

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Starting Result

Theorem: Let $\mathcal{L}$ be a co-complete abelian category with a projective generator $\bar{P}$ ($\mathrm{Hom}(\bar{P}, -)$ is exact and faithful)

If $\mathcal{A} \subseteq \mathcal{L}$, $I: \mathcal{A} \to \mathcal{L}$ exact, is a small abelian subcategory
Then $\exists\, F: \mathcal{A} \to R\text{-Mod}$ fully faithful & exact
(Notice in R-Mod, R is a projective generator)

Strategy:
1) Take $\mathcal{A}$ $\quad \mathcal{A} \overset{I}{\hookrightarrow} \mathcal{B}$ exact, complete & has projective generator
2) Use theorem

Remark: $\mathrm{Hom}(\bar{P}, -)$ is an additive functor
Remark: Not all projectives are generators
<u>Ex</u> $R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$\quad P = \mathbb{Z}/2\mathbb{Z}$ is projective and not a generator

Proof (of theorem): Let $A \in \mathrm{Ob}(\mathcal{A})$ & consider $\displaystyle\coprod_{g \in \mathrm{Hom}(\bar{P}, A)} \bar{P}$

$$\begin{tikzcd}
 & \bar{P} \arrow[d] & \\
\bar{P} \arrow[r, "i_g"] \arrow[dr, bend right, "g"'] & \coprod \bar{P} \arrow[d, "\exists!\, p_A"] & \bar{P} \arrow[l, "i_f"'] \arrow[dl, bend left, "f"] \\
 & A &
\end{tikzcd}$$

$p_A \circ i_g = g$

Remark: p_A is an epimorphism. To show $h_1 \circ p_A = h_2 \circ p_A \Rightarrow h_1 = h_2$, suffices to check $h \circ p_A = 0 \Rightarrow h = 0$, as $h_1 \circ p_A = h_2 \circ p_A \Leftrightarrow (h_1 - h_2) \circ p_A = 0$.

$$p_A : \coprod \bar{P} \longrightarrow A \xrightarrow{\ h\ } B \qquad h \circ p_A = 0$$
$$\Rightarrow h \circ p_A \circ i_g = 0 \quad \forall g : \bar{P} \to A \Rightarrow h \circ g = 0 \quad \forall g : \bar{P} \to A$$

$$\begin{array}{ccc}
\mathrm{Hom}(\bar{P}, A) & \xrightarrow{\mathrm{Hom}(\bar{P}, h)} & \mathrm{Hom}(\bar{P}, B) \\
\rotatebox{90}{$\in$} & & \\
g & \longmapsto & h \circ g = 0
\end{array}$$

By assumption $h \circ g = 0$ $\forall g \in \mathrm{Hom}(\bar{P}, A)$ So $\mathrm{Hom}(\bar{P}, h) = 0$ $\because \bar{P}$ is a generator
$$\Rightarrow \mathrm{Hom}(A, B) \xrightarrow{1-1} \mathrm{Hom}(\mathrm{Hom}(\bar{P}, A), \mathrm{Hom}(\bar{P}, B))$$
$$\begin{array}{ccc}
\rotatebox{90}{$\in$} & & \\
h & \longmapsto & \mathrm{Hom}(\bar{P}, h) \\
0 & \longmapsto & \rotatebox{90}{$=$}\ 0
\end{array}$$
So $h = 0$ $\because \bar{P}$ is a generator
$\Rightarrow p_A$ is an epimorphism

Let $I = \displaystyle\bigcup_{A \in \mathrm{Ob}(A)} \mathrm{Hom}(\bar{P}, A)$, $P := \displaystyle\coprod_{I} \bar{P}$

On A3, we'll see

① P is a projective generator

② $\forall A \in Ob(\mathcal{A})$ $\exists \theta : P \twoheadrightarrow A$ epimorphism

Now we can find a ring R.

$$R := \mathrm{End}\left(\coprod_I P\right)^{op} = \mathrm{End}(P)^{op}$$

Claim: $\exists F$ fully faithful & exact given by $F : \mathcal{A} \to R\text{-Mod}$

$M \in Ob(\mathcal{A}) \mapsto \mathrm{Hom}(P, M)$

Question: Why is F mapping into R-Mod? $R = \mathrm{End}(P)^{op} = \mathrm{Hom}(P,P)^{op} \ni r$

$\psi \in \mathrm{Hom}(P, M)$. What is $r \cdot \psi$?

Answer: $r \cdot \psi := \psi \circ r \in \mathrm{Hom}(P, M)$

$s \cdot (r \cdot \psi) = \cdots = (s * r) \cdot \psi$

Morphisms: $M \xrightarrow{f} N$ $M, N \in Ob(\mathcal{A})$

$\mathrm{Hom}(P,M) \xrightarrow{\mathrm{Hom}(P,f)} \mathrm{Hom}(P,N)$ ⟵ must check this is an R-Mod homomorphism

$g \longmapsto f \circ g$

$\underset{\in R}{r} \cdot (\mathrm{Hom}(P,f)(g)) = r \cdot (f \circ g) = (f \circ g) \circ r = f \circ (g \circ r) = f \circ (r \cdot g)$

$= \mathrm{Hom}(P,f)(r \cdot g)$

Now we must check F is <u>fully</u> faithful & exact

Fact that P is projective $\Rightarrow$ exact

P a generator $\Rightarrow$ faithful

We only need to check that F is full.

If $\alpha : \mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(P,N)$

Then we must show that $\exists f : M \to N$ st $M \xrightarrow{f} N$ $\alpha = \mathrm{Hom}(P,f)$

Now we use the claim $\exists$ epimorphisms $P \xrightarrow{\theta} M$, $P \xrightarrow{\psi} N$

Let $K = \mathrm{Ker}(P \xrightarrow{\theta} M)$

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\theta} M \longrightarrow 0 \qquad \because \mathrm{Hom}(P,-) \text{ is exact}$$

$$0 \longrightarrow \mathrm{Hom}(P,K) \longrightarrow \underset{\substack{\| \\ R}}{\mathrm{Hom}(P,P)} \xrightarrow{\mathrm{Hom}(P,\theta)} \mathrm{Hom}(P,M)$$

as a left R-module if one checks

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(P,K) \arrow[r] & R \arrow[r, "\mathrm{Hom}(P,\theta)"] \arrow[d, dashed, "\exists \alpha'"'] & \mathrm{Hom}(P,M) \arrow[r] \arrow[d, "\alpha"] & 0 \\
& & R \arrow[r, "\mathrm{Hom}(P,\psi)"] & \mathrm{Hom}(P,N) \arrow[r] & 0
\end{tikzcd}$$

$R = \mathrm{Hom}(P,P)$

$$\begin{tikzcd}
& R \ni 1 \arrow[d, "f"] \arrow[dl, dashed, "k"'] & \\
T \arrow[r, two heads, "g"] & U \arrow[r] & 0
\end{tikzcd}$$

$x \longrightarrow f(1)$

$P \xrightarrow{\psi} N \longrightarrow 0$ Fact: R is projective

$\alpha' : R \to R$, $\alpha' = \rho_s$ right multiplication by $s \in R$

Now look at the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d, "s"] & M \arrow[r] & 0 \\ & & P \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

$\rho_s : \mathrm{Hom}(P, s)$

Consider

$$\begin{tikzcd} K \arrow[r] & P \arrow[d, "s"] & & \\ & P \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Claim is 0. Why?

$$\begin{tikzcd} \mathrm{Hom}(P,K) \arrow[r] \arrow[rrr, bend right, "0"'] & R \arrow[r, "\rho_s"] & R \arrow[r] & \mathrm{Hom}(P,N) \end{tikzcd}$$

Now what? $M = \mathrm{coker}(K \to P)$

$$\begin{tikzcd} K \arrow[r] & P \arrow[d, "s"] \arrow[dr, bend right=60, "0"'] & M \arrow[d, "\exists h"] & \\ & P \arrow[r, shift left, "\psi"] \arrow[r, shift right] & N \arrow[r] & 0 \end{tikzcd}$$

by universal property of coker

Now Apply $\mathrm{Hom}(P, -)$ and use fact that $\mathrm{Hom}(P, \theta)$ is an epi.
to conclude $\alpha = \mathrm{Hom}(P, h)$

full!