

# Abelian Categories

A pre-additive category  $\mathcal{C}$  is just a category in which  $\forall A, B \in \text{Ob}(\mathcal{C}), \text{Hom}_{\mathcal{C}}(A, B)$  has the structure of an abelian group

Remark:  $\exists 0_{A,B} : A \rightarrow B, 0_{A,B} + f = f \quad \forall f : A \rightarrow B$  (non-empty)

If  $A, B, C$  objects

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) & \longrightarrow & \text{Hom}_{\mathcal{C}}(A, C) \\ \downarrow \text{q} & \text{Hom}_{\mathcal{C}} & \downarrow \text{f} \\ & \text{q} \cdot \text{f} & \end{array}$$

This map is bilinear, i.e.  $\text{Hom}_{\mathcal{C}}(B, C) \otimes_{\mathbb{Z}} \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$  is a homomorphism

Ex Ring  $R$  Category 1-object  $R$  Morphisms  $\varphi_r, r \in R, \varphi_r : R \rightarrow R$   
 $\varphi_0 = \varphi_{1,R}, \varphi_r + \varphi_s = \varphi_{r+s}, (\varphi_r \circ \varphi_s) \circ \varphi_t = \varphi_r \circ (\varphi_s \circ \varphi_t)$   
 $\varphi_r(x) = rx$

So this category is preadditive.

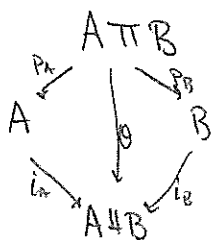
Remark: If  $\mathcal{C}, \mathcal{D}$  are pre-additive categories,  $A, B \in \text{Ob}(\mathcal{C})$   $F : \mathcal{C} \rightarrow \mathcal{D}$  is called additive if the map  $f \mapsto F(f)$  gives a homomorphism from  $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(FA, FB)$ .

Def) A pre-additive category is additive if all finite (including empty) products & coproducts exist

Remark:  $\mathcal{C}$  additive,  $A, B \in \text{Ob}(\mathcal{C})$

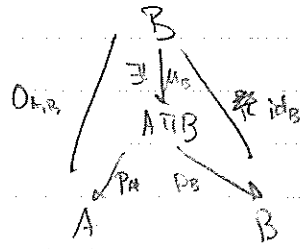
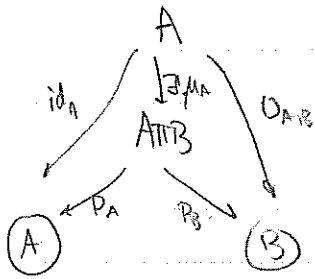
$$\begin{array}{ccc} A \amalg B & \cong & A \amalg B \\ \text{"u"} & & \text{"u"} \\ A \times B & & A \oplus B \end{array}$$

Proof:



$\theta$  part doesn't commute

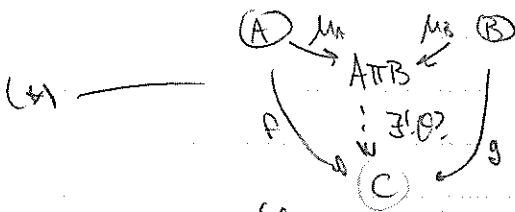
$\theta \cong$  should be  $i_A \circ p_A + i_B \circ p_B$



$$id_A = p_A \circ \mu_A \quad o = p_A \circ \mu_B$$

$$o = p_B \circ \mu_A \quad p_B = p_B \circ \mu_B$$

Claim  $A\pi B$  is coproduct



What should  $\theta$  be?

$\theta$  should be  $f \circ p_A + g \circ p_B$

Let's check that this  $\theta$  works!

Must show  $f = \theta \circ \mu_A$ ,  $g = \theta \circ \mu_B$

$$\theta \circ \mu_A = (f \circ p_A + g \circ p_B) \circ \mu_A$$

$$= f \circ p_A \circ \mu_A + g \circ p_B \circ \mu_A = f \circ id_A + g \circ o = f$$

similarly for  $g$

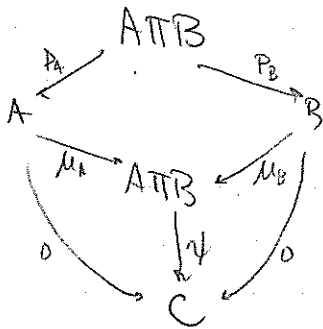
Let's show  $\theta$  is unique.

Suppose  $\theta$  &  $\theta'$  both make diagram (\*) commute

$$\theta \circ \mu_A = f \quad \theta' \circ \mu_A = f$$

$$\theta \circ \mu_B = g \quad \theta' \circ \mu_B = g$$

Let  $\psi = \theta - \theta'$ ,  $\psi \circ \mu_A = 0$ ,  $\psi \circ \mu_B = 0$



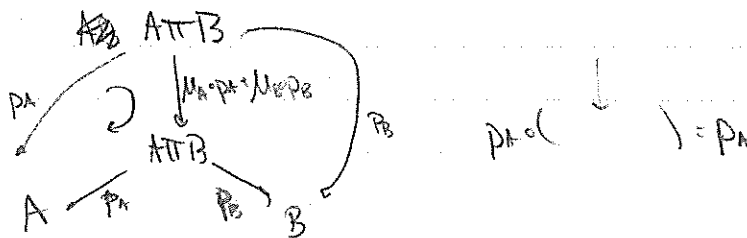
Claim:  $\mu_A \circ p_A + \mu_B \circ p_B = id_{A\pi B}$

Once we have the claim,

$$\psi = \psi \circ id_{A\pi B} = \psi \circ (\mu_A \circ p_A + \mu_B \circ p_B)$$

$$= \underbrace{\psi \circ \mu_A \circ p_A}_0 + \underbrace{\psi \circ \mu_B \circ p_B}_0 = 0$$

Why?

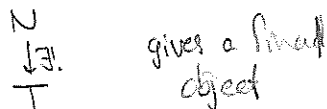
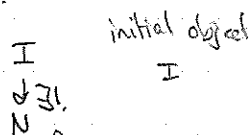


But  $id_{A\pi B}$  also works in place of  $\mu_A \circ p_A + \mu_B \circ p_B$ , so uniqueness gives the claim.

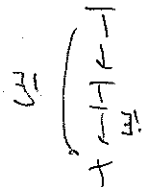
Remark: We also have a zero object

Why? Empty coproduct

Empty product



Claim:  $I \cong T$ . Henceforth we'll write  $0$  for the zero object.



Notice if  $f: A \rightarrow B$   $A \xrightarrow[\text{on } B]{f} B$  limit = Equalizer

Intuitively " $\{x \in A, f(x) = 0\}$ "

If the equalizer  $A \xrightarrow[\text{on } B]{f} B$  exists, we call it the kernel of  $f$ .

Similarly,  $\text{coker}(f) := \lim_{\rightarrow} (A \xrightarrow[\text{on } B]{f} B)$  if it exists.

An additive category in which kernels & cokernels exist is called pre-abelian.

Abelian Categories

$$A \rightarrow B$$

Def: A map  $f: A \rightarrow B$  is called monomorphism "~~A~~" if  $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2$ .

$f: A \rightarrow B$  is an epimorphism if  $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$

monomorphism + epimorphism  $\not\Rightarrow$  isomorphism

Eg.  $\mathbb{R}$  - category of commutative rings  $\mathbb{Z} \xrightarrow{i} \mathbb{Q} \begin{matrix} \xrightarrow{h_1} \\ \xrightarrow{h_2} \end{matrix} \mathbb{R}$   $i$  is a monomorphism

Claim  $h_1 \circ i = h_2 \circ i \Rightarrow h_1 = h_2$

$$h_1(n) = h_2(n) \quad \forall n \in \mathbb{Z}$$

$$h_1(b) h_1\left(\frac{1}{b}\right) = h_1(1) = 1 \quad \forall b \in \mathbb{Z} \setminus \{0\}$$

$$h_2(b) h_2\left(\frac{1}{b}\right) = 1$$

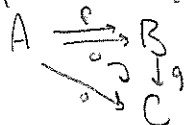
$$h_1\left(\frac{a}{b}\right) = h_1(a) h_1\left(\frac{1}{b}\right) = h_2(a) h_2\left(\frac{1}{b}\right) = h_2\left(\frac{a}{b}\right) \Rightarrow h_1 = h_2$$

An abelian category is one in which every monomorphism & epimorphism is normal.

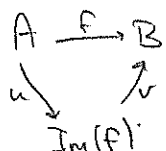
$A \xrightarrow{f} B$  monomorphism

$f$  is Normal if  $f$  is a kernel

Similarly  $B \xrightarrow{g} C$  epimorphism  $B$  normal if  $g$  is a cokernel



Exercise: This implies



$u$  is an epimorphism  
 $v$  is a monomorphism

$$A \xrightarrow{f} B \rightarrow B/\text{Im}(f)$$

Coker(f)

$$\text{Im}(f) = \ker(\text{Coker}(f))$$

Example  $R$  ring with 1, associative, not necessarily commutative

$R\text{-Mod}$  = Category of left  $R$ -modules

is an abelian ~~category~~ category

Remark: In  $R\text{-Mod}$

$$\ker(f) \xrightarrow{i} M \xrightarrow{f} N \text{ is a monomorphism } \Leftrightarrow f \text{ is 1-1}$$

$$f \circ i = f \circ 0 \quad f \text{ monomorphism} \Rightarrow i = 0 \Rightarrow \ker(f) = (0)$$

Similarly, epimorphisms are onto

Division

Mitchell's embedding lemma: Let  $A$  be a small abelian category. Then

$F: A \rightarrow R\text{-Mod}$ ,  $R$  ring  $F$  is full, faithful, exact

$A, B \in \text{ob}(A)$

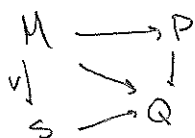
$$F: \text{Hom}_A(A, B) \rightarrow \text{Hom}_R(F(A), F(B))$$

-full:  $F$  is onto  $\forall A, B$

-faithful:  $F$  is 1-1  $\forall A, B$

exact: If  $F$  is additive & if  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is exact in  $A$

$$\Rightarrow 0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \rightarrow 0 \text{ is exact in } R\text{-Mod}$$



"let's try to outthink this theorem."  
 "I know you can't outthink a theorem."

What is exactness?

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Then what does it mean to say this is exact at B?

1)  $g \circ f = 0$  &

2) the canonical map  $\tilde{f}: \text{Im}(f) \rightarrow \ker(g)$  is an isomorphism

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow u & \searrow \tilde{f} & \downarrow v \\ & \text{Im}(f) & \end{array} \quad \text{Im}(f) = \ker(\text{Coker}(f))$$

$$\begin{array}{ccc} \exists! \theta & \rightarrow & \text{Im}(f) = \ker(\pi) \\ & \searrow & \downarrow i \\ A & \xrightarrow{f} & B \\ & \searrow \theta & \downarrow \pi \\ & & \text{Coker}(f) \end{array} \quad \text{by universal property of kernel}$$

$$\begin{array}{ccccc} & \theta & & & \\ & \nearrow & \text{Im}(f) & \searrow i & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow \theta & & \searrow g & \\ & & & & \text{Coker}(f) \end{array} \quad \begin{array}{l} \theta \text{ epimorphism} \\ i \text{ monomorphism} \end{array}$$

$$0 = g \circ f \Rightarrow g \circ i \circ \theta = 0 \Rightarrow g \circ i = 0 \Rightarrow \theta \text{ is epi}$$

$$\begin{array}{ccc} \text{Im}(f) & \xrightarrow{\tilde{f}} & \ker(g) \\ \downarrow i & \swarrow & \\ B & \xrightarrow{g} & C \end{array}$$

Next Remark  $0 \rightarrow A \xrightarrow{f} B$  is exact at A  $\Leftrightarrow f$  is a monomorphism  
 $B \xrightarrow{g} C \rightarrow 0$  is exact at C  $\Leftrightarrow g$  is an epimorphism  
 $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$

Mitchell's Theorem

A small abelian category  
 $\downarrow$   
 $R\text{-Mod}$

If we start with  $R\text{-Mod}$ , can we "recover"  $R$ .

Remark: If  $\mathcal{A}$  is an abelian category  $A \in \text{Ob}(\mathcal{A})$   
 $\text{Hom}_{\mathcal{A}}(A, A) = \text{End}_{\mathcal{A}}(A)$  is a ring under  $\circ$

In  $R\text{-Mod}$   $R$  as a left  $R$ -Module

$$\text{End}_R(R) \cong R^{\text{op}} \quad s * r = r \cdot s$$

What's the isomorphism  $\psi: R \rightarrow R$

$\psi$  is determined by  $\psi(1)$ . If  $\psi(1) = s$ ,  $\psi(r) = r \cdot \psi(1) = r \cdot s$

So  $\psi = \Phi_s$  for some  $s \in R$  where  $\Phi_s(x) = x \cdot s$

$$\text{End}_R(R) \cong \{\psi_s; s \in R\} \cong R^{\text{op}}$$

$$\psi_s \circ \psi_r(x) = \psi_s(xr) = xr \cdot s = \psi_{rs}(x) \quad \psi_s \circ \psi_r = \psi_{rs}$$

$$\exists R \cong S \quad R\text{-Mod} \cong S\text{-Mod}$$

$$\text{Ex } R\text{-Mod} \cong M_n(R)\text{-Mod}$$

$$R^n \rightsquigarrow \text{End}_R(R^n) \cong M_n(R^{\text{op}}) \quad \text{So } \text{End}_R(R^n)^{\text{op}} \cong M_n(R)$$

Fix Notion ~~is~~ close to freeness

Let  $\mathcal{A}$  be an abelian category,  $M \in \text{Ob}(\mathcal{A})$

$$\text{Hom}(M, -): \mathcal{A} \rightarrow \text{Ab}$$

$$B \mapsto \text{Hom}_{\mathcal{A}}(M, B)$$

We say that  $M$  is a projective object of  $\mathcal{A}$  if the functor  $\text{Hom}(M, -)$  is exact

What are projectives in  $R\text{-Mod}$ ?  $\text{Hom}(P, -)$  is left exact check  $\forall P$

When is  $\text{Hom}(P, -)$  right exact?

$$M \xrightarrow{f} N \rightarrow 0 \text{ exact} \Rightarrow \text{Hom}(P, M) \xrightarrow{f_*} \text{Hom}(P, N) \rightarrow 0 \text{ is exact}$$

$$M \xrightarrow{f} N \rightarrow 0$$

$$\begin{array}{ccc} \uparrow & \uparrow \varphi & \\ \exists \psi & \cdot & P \end{array}$$

Remark:  $P$  projective  $\Rightarrow \exists Q$  st  $P \oplus Q \cong R^I$

$$\text{Idea! } R^I \twoheadrightarrow P \rightarrow 0$$

$$\begin{array}{ccc} \uparrow & \uparrow \text{id} & \\ s & \cdot & P \end{array}$$

$$\Leftrightarrow Q \rightarrow R^I \xrightarrow{\pi} P \rightarrow 0 \Rightarrow R^I \cong Q \oplus P$$

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Starting Result

Theorem: Let  $\mathcal{A}$  be a  $\infty$ -complete abelian category with a projective generator  $P$  ( $\text{Hom}(P, -)$  is exact and faithful)

If  $\mathcal{A} \subseteq \mathcal{L}$ ,  $I: \mathcal{A} \rightarrow \mathcal{L}$  exact, is a small abelian subcategory  
 Then  $\exists F: \mathcal{A} \rightarrow R\text{-Mod}$  fully faithful & exact  
 (Notice in  $R\text{-Mod}$ ,  $R$  is a projective generator)

Strategy:

- 1) Take  $\mathcal{A} \xrightarrow{I} \mathcal{B}$  exact, complete & has projective generator
- 2) Use theorem

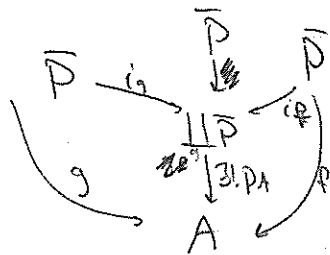
Remark:  $\text{Hom}(\bar{P}, -)$  is an additive functor

Remark: Not all projectives are generators

Ex  $R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

$P = \mathbb{Z}/2\mathbb{Z}$  is projective and not a generator

Proof (of theorem): Let  $A \in \text{Ob}(\mathcal{A})$  & consider



$$\coprod_{g \in \text{Hom}(\bar{P}, A)} \bar{P}$$

$$p_A \circ i_g = g$$

Remark:  $p_A$  is an epimorphism. To show  $h_1 \circ p_A = h_2 \circ p_A \Rightarrow h_1 = h_2$ ,  
 suffices to check  $h \circ p_A = 0 \Rightarrow h = 0$ , as  $h_1 \circ p_A = h_2 \circ p_A \Leftrightarrow (h_1 - h_2) \circ p_A = 0$ .

$$p_A: \coprod \bar{P} \rightarrow A \xrightarrow{h} B \quad h \circ p_A = 0$$

$$\Rightarrow h \circ p_A \circ i_g = 0 \quad \forall g: \bar{P} \rightarrow A \Rightarrow h \circ g = 0 \quad \forall g: \bar{P} \rightarrow A$$

$$\begin{array}{ccc} \text{Hom}(\bar{P}, A) & \xrightarrow{\text{Hom}(\bar{P}, h)} & \text{Hom}(\bar{P}, B) \\ \downarrow g & \longleftarrow & \downarrow h \circ g = 0 \end{array}$$

By assumption,  $h \circ g = 0 \quad \forall g \in \text{Hom}(\bar{P}, A)$  So  $\text{Hom}(\bar{P}, h) = 0 \quad \therefore \bar{P}$  is a generator

$$\Rightarrow \text{Hom}(\text{Hom}(\bar{P}, B)) \xrightarrow{i^{-1}} \text{Hom}(\text{Hom}(\bar{P}, A), \text{Hom}(\bar{P}, B))$$

$$\begin{array}{ccc} \downarrow h & \longleftarrow & \downarrow \text{Hom}(\bar{P}, h) \\ 0 & \longleftarrow & 0 \end{array}$$

So  $h = 0 \quad \therefore \bar{P}$  is a generator  
 $\Rightarrow p_A$  is an epimorphism

$$\text{Let } \mathcal{I} = \bigcup_{A \in \text{Ob}(\mathcal{A})} \text{Hom}(\bar{P}, A), \quad \bar{P} := \coprod_{\mathcal{I}} \bar{P}$$

On A3, we'll see

① P is a projective generator

②  $\forall A \in \text{Ob}(A) \exists \theta: P \rightarrow A$  -epimorphism

Now we can find a ring R.

$$R := \text{End} \left( \begin{array}{c} \text{H} \\ \text{I} \end{array} P \right)^{\text{op}} = \text{End}(P)^{\text{op}}$$

Claim:  $\exists F$  fully faithful & exact given by  $F: A \rightarrow R\text{-Mod}$

$$\text{Mod}(A) \mapsto \text{Hom}(P, M)$$

Question: Why is F mapping into R-Mod?  $R = \text{End}(P)^{\text{op}} = \text{Hom}(P, P)^{\text{op}}$  or

$\forall \psi \in \text{Hom}(P, M)$ . What is  $r \cdot \psi$ ?

Answer:  $r \cdot \psi = \psi \circ r \in \text{Hom}(P, M)$

$$s \cdot (r \cdot \psi) = \dots = (s \cdot r) \cdot \psi$$

Morphisms:  $M \xrightarrow{f} N \quad M, N \in \text{Ob}(A)$

$$\text{Hom}(P, M) \xrightarrow{\text{Hom}(P, f)} \text{Hom}(P, N) \quad \leftarrow \text{must check this is an } R\text{-Mod homomorphism}$$

$$\begin{array}{c} \uparrow \\ \text{Hom}(P, M) \xrightarrow{f \circ \_} \text{Hom}(P, N) \\ \uparrow \\ \text{Hom}(P, M) \end{array} \quad \begin{array}{c} \text{Hom}(P, f) \\ \text{Hom}(P, f) \end{array}$$

$$r \cdot (\text{Hom}(P, f)(g)) = r \cdot (f \circ g) = (f \circ g) \circ r = f \circ (g \circ r) = f \circ (r \cdot g) = \text{Hom}(P, f)(r \cdot g)$$

Now we must check F is fully faithful & exact

Fact that P is projective  $\Rightarrow$  exact

P a generator  $\Rightarrow$  faithful

We only need to check that F is full.

If  $g: \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$

Then we must show that  $\exists f: M \rightarrow N$  st  $M \xrightarrow{f} N \quad g = \text{Hom}(P, f)$

Now we use the claim  $\exists$  epimorphisms  $P \xrightarrow{\theta} M, P \xrightarrow{\psi} N$

Let  $K = \text{Ker}(P \xrightarrow{\theta} M)$

$$0 \rightarrow K \rightarrow P \xrightarrow{\theta} M \rightarrow 0 \quad \because \text{Hom}(P, -) \text{ is exact}$$

$$0 \rightarrow \text{Hom}(P, K) \rightarrow \text{Hom}(P, P) \xrightarrow{\text{Hom}(P, \theta)} \text{Hom}(P, M)$$

R as a left R-module  
if one checks

$$0 \rightarrow \text{Hom}(P, K) \rightarrow R \xrightarrow{\text{Hom}(P, \theta)} \text{Hom}(P, M) \rightarrow 0$$

$$\begin{array}{ccc} \exists \alpha' & & \downarrow g \\ R & \xrightarrow{\text{Hom}(P, \psi)} & \text{Hom}(P, N) \rightarrow 0 \\ \text{Hom}(P, P) & & \end{array}$$

$P \xrightarrow{\psi} N \rightarrow 0$  Fact: R is projective

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow f & & \\ T & \xrightarrow{g} & U & \rightarrow & 0 \\ x & \longrightarrow & f(x) & & \end{array}$$



$\alpha: R \rightarrow R$ ,  $\alpha = \rho_s$  right multiplication by  $s \in R$

Now look at the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M \longrightarrow 0 \\ & & & & \downarrow s & & \\ & & & & P & \longrightarrow & N \longrightarrow 0 \end{array}$$

$\rho_s = \text{Hom}(P, s)$

Consider  $\begin{array}{c} K \longrightarrow P \\ \downarrow s \\ P \longrightarrow N \longrightarrow 0 \end{array}$

Claim is 0. Why?

$$\text{Hom}(P, K) \longrightarrow R \xrightarrow{\rho_s} R \longrightarrow \text{Hom}(P, N)$$

(A curved arrow labeled 0 points from the first R to the last Hom(P, N))

Now what?  $M = \text{coker}(K \rightarrow P)$

$$\begin{array}{ccc} K & \longrightarrow & P \\ & & \downarrow s \\ & & P \end{array} \xrightarrow{\quad} \begin{array}{ccc} M & & \\ \exists! h & \downarrow & \\ N & \longrightarrow & 0 \end{array}$$

(A curved arrow labeled 0 points from the bottom P to the N)

by universal property of coker

Now Apply  $\text{Hom}(P, -)$  and use fact that  $\text{Hom}(P, \theta)$  is an epi. to conclude  $\alpha = \text{Hom}(P, h)$

↑  
full!