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Govorov-Lazard Theorem

Recall that an R-module M is <u>flat</u> if $0 \to N' \xrightarrow{f} N$ exact $\Rightarrow 0 \to N' \otimes M \to N \otimes M$ exact

<u>projective</u>

$Hom_R(P, -)$ exact $\iff$ P projective

$Hom_R(-, I)$ exact $\iff$ I injective

<u>Eg</u> Free modules are flat

G-L Thm: Let R be a commutative ring & let M be an R-module. Then M is flat if and only if M is a <u>filtered colimit</u> of free modules.

Filtered Categories

$\mathcal{B}$ small category. We'll say that $\mathcal{B}$ is <u>filtered</u> if

① $B_1, B_2 \in Ob(\mathcal{B}) \Rightarrow \exists B$

$$\begin{tikzcd} B_1 \arrow[dr, "f"] & \\ & B \\ B_2 \arrow[ur, "g"'] & \end{tikzcd}$$

② If $\exists f \in Hom(B', B_1), g \in Hom(B', B_2)$

$$\begin{tikzcd} & B_1 \arrow[dr, "u"] & \\ B' \arrow[ur, "f"] \arrow[dr, "g"'] & \circlearrowleft & B'' \\ & B_2 \arrow[ur, "v"'] & \end{tikzcd}$$

Then $\exists B''$ & $u: B_1 \to B''$, $v: B_2 \to B''$

If $F: \mathcal{B} \to \mathcal{A}$ is a diagram & $\mathcal{B}$ is filtered, we'll say $\varinjlim F$ is a filtered colimit.

Filtered limits in R-Mod. If $\mathcal{B}$ is a filtered subcategory of R-Mod. Then what is $\varinjlim \mathcal{B}$?

Concrete descriptions

$$\varinjlim \mathcal{B} = \coprod_{M \in Ob(\mathcal{B})} M \Big/ \sim$$

What is $\sim$? $x \in M$, $y \in M'$ Then $x \sim y \iff \exists M''$ & $f: M \to M''$ & $g: M' \to M''$ st $f(x) = g(y)$.

Remarks:

① $\sim$ is an equivalence relation

② $\coprod_{M\in Ob(\mathcal{B})} M/\sim$ is an R-Module

Proof. ① Transitivity:

$$\begin{tikzcd}
x\in M \arrow[dr, "f"] & & \\
y\in M' \arrow[r, "g"] \arrow[dr, "h"'] & P \arrow[r, "u"] & Q \\
z\in M'' \arrow[r, "k"'] & P' \arrow[ur, "v"'] &
\end{tikzcd}$$

② If $x, y \in \coprod M$, $x\in M_1$, $y \in M_2$ how should we define $x+y$?

Since $\mathcal{B}$ is filtered $\exists$ $M_1 \xrightarrow{f} N$, $M_2 \xrightarrow{g} N$. Define $x+y := f(x)+g(y)$ mod $\sim$

Must show well-defined

$M_1 \ni x \sim x' \in M_1'$

$M_2 \ni y \sim y' \in M_2'$

$$\begin{tikzcd}
 & P \ni x'' & & \\
M_1 \arrow[r, "f"] \arrow[ur] & N & M_1' \arrow[ul] \arrow[r, "f'"] & N' \\
M_2 \arrow[ur, "g"'] \arrow[dr] & & M_2' \arrow[dl] \arrow[ur, "g'"'] & \\
 & Q \ni y'' & &
\end{tikzcd}$$

Guess $x+y \sim x'+y' \sim x''+y''$ just do some diagrams

$$\begin{tikzcd}
M^{\in Ob(\mathcal{B})} \arrow[dr] \arrow[ddr, bend right, "\varphi_M"'] & \varinjlim \mathcal{B} \arrow[d] \\
 & \coprod_{M\in Ob(\mathcal{B})} M/\sim \arrow[d, "\exists!"] \\
 & F
\end{tikzcd}$$

Suppose that (F, φ_M) is a cocone over $\mathcal{B}$

$$\begin{tikzcd}
 & M_1 \arrow[dr] \arrow[ddd, bend right] & \\
M' \arrow[ur, "f"] \arrow[r, "g"'] \arrow[ddr, bend right] & M_2 \arrow[r] \arrow[d] & M'' \arrow[ddl, bend left] \\
 & \coprod M/\sim & \\
 & F &
\end{tikzcd}$$

We have maps

$$M \xrightarrow{i_M} \bigcup M/\sim$$
$$m \longmapsto [m]_\sim \quad \text{well-defined}$$

If $x \sim y$

$$\begin{tikzcd} x \in M \arrow[dr, "u"] & \\ & M'' \\ y \in M' \arrow[ur, "v"'] & \end{tikzcd} \quad u(x) = v(y)$$

$$\begin{tikzcd}
M \arrow[rr, "u"] \arrow[dd, bend right, "\varphi_M"'] & & M'' \arrow[ddl, bend left, "\varphi_{M''}"] \\
M' \arrow[urr, "v"] \arrow[d, "\varphi_{M'}"'] \arrow[r] & \varinjlim \mathcal{B} \arrow[dl, "\exists!"] & \\
F & &
\end{tikzcd}$$

$$\varphi_M(x) = \varphi_{M''} \circ u(x)$$
$$= \varphi_{M''} \circ v(y) = \varphi_{M'}(y)$$

We'll prove one direction of G-L thm

Namely $\varinjlim \mathcal{U}_i$ $i \in I$ $\mathcal{U}_i$ free filtered subcategory of R-mod whose objects are free $\Rightarrow$ $\varinjlim \mathcal{U}_i$ is flat

Idea: Suppose that $0 \longrightarrow N' \xrightarrow{f} N$ is exact & $M = \varinjlim \mathcal{U}_i$

Goal: $0 \longrightarrow M \otimes N' \xrightarrow{id \otimes f} M \otimes N$ is exact

Let $F : \mathcal{B} \longrightarrow R\text{-Mod}$ $\quad Q \longmapsto Q \otimes N'$

$\quad G : \mathcal{B} \longrightarrow R\text{-Mod}$ $\quad Q \longmapsto Q \otimes N$

key idea: $\alpha : F \to G$ natural transformation $U \in Ob(\mathcal{B})$

$$F(U) \xrightarrow{\alpha_U} G(U)$$
$$U \otimes N' \xrightarrow{id \otimes f} U \otimes N$$

$$U \xrightarrow{h} U'$$

$$u \otimes n' \longrightarrow u \otimes f(n')$$

$$\begin{tikzcd}
F(U) \arrow[r, "\alpha_U"] \arrow[d, "F(h)"] & G(U) \arrow[d, "G(h)"] \\
F(U') \arrow[r, "\alpha_{U'}"] & G(U')
\end{tikzcd}$$

Remark: α_U is an injection

$$h(u) \otimes f(n')$$
$$h(u) \otimes n' \longrightarrow h(u) \otimes f(n')$$

Now here is the proof $\quad M = \varinjlim \mathcal{U}_i$

$$M \otimes N' = (\varinjlim \mathcal{U}_i) \otimes N'$$
$$\downarrow \text{is}$$
$$\varinjlim (\mathcal{U}_i \otimes N') \xrightarrow{\varinjlim} \varinjlim (\mathcal{U}_i \otimes N)$$
$$\cong$$
$$(\varinjlim \mathcal{U}_i) \otimes N = M \otimes N$$

Notice to see
$$\varinjlim(U_i \otimes N') \xrightarrow{\lim} \varinjlim(U_j \otimes N) \text{ is 1-1}$$
$$\bigsqcup U_i \otimes N'/\sim \xrightarrow{id \otimes f} \bigsqcup U_i \otimes N/\sim$$

Why is this injective? (It is well-defined!)
Suppose $x \in \bigsqcup U_i \otimes N'/\sim$ gets sent to 0, ie $h(x) \sim 0$

$h(x) \in U_i \otimes N \xrightarrow{G(\psi) = \theta} U_j \otimes N$

$$h(x) \xrightarrow{\theta} 0$$

$$\begin{tikzcd}
x \arrow[rrr, bend left] \arrow[ddd, bend right] & U_i \otimes N' \arrow[r, "\alpha_{u_i}"] \arrow[d, "F(\psi)"] & U_i \otimes N \arrow[d, "G(\psi)"] & h(x) \arrow[ddd, bend left] \\
& U_j \otimes N' \arrow[r, "\alpha_{u_j}"] & U_j \otimes N & \\
& & & \\
y \arrow[rrr, bend right] & & & 0
\end{tikzcd}$$

$\because \alpha_{u_j}$ is 1-1 $\Rightarrow y = 0$

$\Rightarrow x \sim 0$ so $M \otimes N' \xrightarrow{id \otimes f} M \otimes N$ is 1-1 $\Rightarrow M$ is flat