

Covorov-Lazard Theorem

Recall that an R -module M is flat if $0 \rightarrow N \xrightarrow{f} N \text{ exact} \Rightarrow 0 \rightarrow N \otimes M \xrightarrow{f \otimes 1} N \otimes M \text{ exact}$

projective

$\text{Hom}_R(P, -) \text{ exact} \Leftrightarrow P \text{ projective}$

$\text{Hom}_R(-, I) \text{ exact} \Leftrightarrow I \text{ injective}$

Eg Free modules are flat

G-L Thm. Let R be a commutative ring & let M be an R -module. Then M is flat if and only if M is a filtered colimit of free modules.

Filtered Categories

\mathcal{B} small category. We'll say that \mathcal{B} is Filtered if

① $B_1, B_2 \in \text{Ob}(\mathcal{B}) \Rightarrow \exists B$

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B \\ & \searrow g & \uparrow \\ & & B_2 \end{array}$$

② If $f \in \text{Hom}(B, B_1), g \in \text{Hom}(B, B_2)$

$$\begin{array}{ccccc} & & B_1 & \xrightarrow{u} & \\ & f \searrow & & & \\ B & & & C & \\ & \swarrow g & & & \\ & & B_2 & \xrightarrow{v} & B'' \end{array}$$

Then $\exists B''$ & $u: B_1 \rightarrow B'', v: B_2 \rightarrow B''$

If $F: \mathcal{B} \rightarrow \mathcal{A}$ is a diagram & \mathcal{B} is filtered, we'll say $\varinjlim F$ is a filtered colimit.

Filtered limits in $R\text{-Mod}$. If \mathcal{B} is a filtered subcategory of $R\text{-Mod}$.

Then what is $\varinjlim \mathcal{B}$?

Concrete descriptions

$$\varinjlim \mathcal{B} = \bigsqcup_{\text{Mod}(\mathcal{B})} M / \sim$$

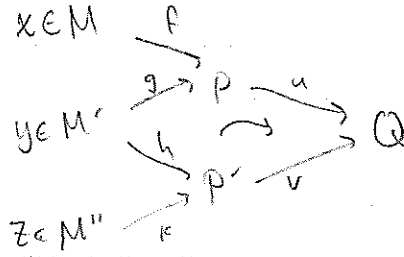
What is \sim ? $x \in M, y \in M'$. Then $x \sim y \Leftrightarrow \exists M''$ & $f: M \rightarrow M''$
& $g: M' \rightarrow M''$
st $f(x) = g(y)$

Remarks:

① \sim is an equivalence relation

② $\coprod_{M \in \mathcal{B}} M/\sim$ is an \mathbb{R} -Module

Proof. ① Transitivity:



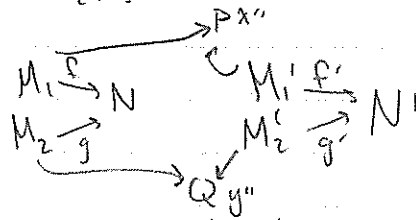
② If $x, y \in \coprod M$, $x \in M_1, y \in M_2$ how should we define $x+y$?

Since \mathcal{B} is filtered $\exists M_1 \xrightarrow{f} N$ Define $x+y := f(x) + g(y) \pmod{\sim}$

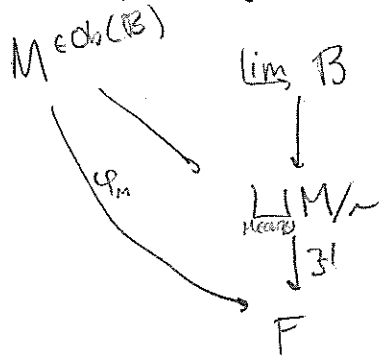
Must show well-defined

$M_1 \ni x \sim x' \in M_1'$

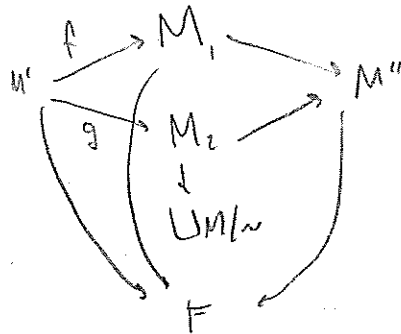
$M_2 \ni y \sim y' \in M_2'$



Guess $x+y \sim x'+y' \sim x''+y''$ just do some diagrams



Suppose that (F, φ_M) is a colimit over \mathcal{B}



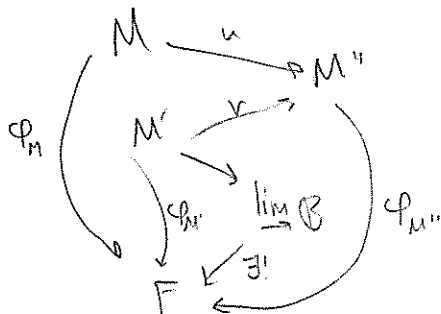
We have maps

$$M \xrightarrow{u} \bigcup M / \sim$$

$$m \mapsto [m] \sim \text{well-defined}$$

If $x \sim y$

$$\begin{array}{ccc} x \in M & \xrightarrow{u} & M'' \\ y \in M' & \xrightarrow{v} & M'' \end{array} \quad u(x) = v(y)$$



$$\begin{aligned} \phi_M(x) &= \phi_{M''} \circ \phi_M'(x) \\ &= \phi_{M''} \circ v(y) = \phi_{M''}(y) \end{aligned}$$

We'll prove one direction of G-L thm

Namely $\varinjlim U_i$ $i \in I$ U_i free filtered subcategory of $R\text{-mod}$ whose objects are free $\Rightarrow \varinjlim U_i$ is flat

Idea: Suppose that $0 \rightarrow N' \xrightarrow{f} N$ is exact & $M = \varinjlim U_i$

Goal: $0 \rightarrow M \otimes N' \xrightarrow{\text{id} \otimes f} M \otimes N$ is exact

Let $F: \mathcal{B} \rightarrow R\text{-Mod}$ $U \mapsto U \otimes N'$

$G: \mathcal{B} \rightarrow R\text{-Mod}$ $U \mapsto U \otimes N$

key idea: $\alpha: F \rightarrow G$ natural transformation $U \in \text{Ob}(\mathcal{B})$

$$F(U) \xrightarrow{\alpha_U} G(U)$$

$$U \otimes N' \xrightarrow{\text{id} \otimes f} U \otimes N$$

$$U \xrightarrow{h} U'$$

$$u \otimes n' \mapsto u \otimes f(n')$$

$$F(U) \xrightarrow{\alpha_U} G(U)$$

$$\downarrow F(h)$$

$$\downarrow G(h)$$

$$F(U') \xrightarrow{\alpha_{U'}} G(U')$$

$$h(n') \otimes f(n')$$

$$h(n') \otimes n'$$

$$h(n') \otimes f(n')$$

Remark: α_U is an injection

Now here is the proof $M = \varinjlim U_i$

$$M \otimes N' = \left(\varinjlim U_i \right) \otimes N'$$

$$\varinjlim (U_i \otimes N') \xrightarrow{\varinjlim} \varinjlim (U_i \otimes N)$$

$$\left(\varinjlim U_i \right) \otimes N = M \otimes N$$

Notice to see

$$\varinjlim (U_i \otimes N') \xrightarrow[\cong]{\varinjlim} \varinjlim (U_i \otimes N)$$

$$\sqcup U_i \otimes N' / \sim \xrightarrow{\text{id}} \sqcup U_i \otimes N / \sim$$

Why is this injective? (if is well-defined!)

Suppose $x \in \sqcup U_i \otimes N' / \sim$ gets sent to 0, i.e. $h(x) = 0$

$$h(x) \in U_j \otimes N \xrightarrow{\alpha_j} U_j \otimes N$$

$$\begin{array}{ccc} x & \xrightarrow{h(x)} & 0 \\ \downarrow \alpha_i & \searrow & \downarrow \alpha_j \\ U_i \otimes N' & \xrightarrow{\alpha_i} & U_i \otimes N \\ \downarrow F(\phi) & & \downarrow G(\psi) \\ y & \xrightarrow{\alpha_j} & U_j \otimes N' \\ \downarrow & \searrow & \downarrow \\ & & 0 \end{array}$$

$$\alpha_j \text{ is } \cong \Rightarrow y = 0$$

$$\Rightarrow x = 0 \text{ so } M \otimes N' \xrightarrow{\text{id}} M \otimes N \text{ is } \cong \Rightarrow M \otimes N \text{ is flat}$$