Initial & Terminal objects

$I \in Ob(\mathcal{C})$ is an *initial* object if $\forall C \in Ob(\mathcal{C})\ \exists!\ f: I \to C$
T is a *terminal* object if $\exists!\ g: C \to T\ \forall$ object C

Ex *Sets*
$I = \emptyset$
$T = \{\cdot\}$

If they exist, initial & terminal objects are unique up to unique isomorphism.
I_1, I_2 initial

$$\begin{tikzcd} I_1 \arrow[r, bend left, "\exists i_1"] & I_2 \arrow[l, bend left, "\exists i_2"] \end{tikzcd} \qquad i_2 \circ i_1 : I_1 \xrightarrow{id_{I_1}} I_1 \quad \text{So } i_2 \circ i_1 = Id_{I_1}$$

unique map

Eg Rings
$I = \mathbb{Z}$
$T = 0_R$

$$\mathbb{Z} \xrightarrow{\exists!} R$$
$$1 \longmapsto 1_R$$

$R \longrightarrow 0_R$
$R \longrightarrow R/R$

Ab
$I = (0)$
$T = (0)$ } Zero objects

Field* no initial or terminal objects
*ignoring F_1

Limits & Colimits

$\varinjlim$ Colimits
$\varprojlim$ Limits

Let $\mathcal{C}$ be a category & let $\mathcal{B}$ be a category (Almost always $\mathcal{B}$ will be small & $\mathcal{B} \subseteq \mathcal{C}$ not necessarily full). Then a *diagram* based on $\mathcal{B}$ is a functor $F: \mathcal{B} \to \mathcal{C}$ (F will often be inclusion functor)

$$\begin{tikzcd}[column sep=small] & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr, bend left] & & \circ \end{tikzcd}$$

A diagram is <u>small</u> if $\mathcal{B}$ is a small category

$F: \mathcal{B} \to \mathcal{C}$ A <u>cone</u> to F is an object $N \in Ob(\mathcal{C})$ & a family of morphisms $\varphi_B : N \to FB$ $\forall B \in Ob(\mathcal{B})$

$$\begin{tikzcd} & N \arrow[dl, "\varphi_i"'] \arrow[d, "\varphi_j"] \arrow[drr, "\varphi_k"] & & \\ FB_i \arrow[r, "F(f)"'] & FB_j \arrow[l, dotted] & \cdots & FB_k \end{tikzcd} \qquad i,j,k \in I$$

$f: B_i \xrightarrow[\mathcal{B}]{} B_j$ then $F(f) \circ \varphi_i = \varphi_j$ $\forall$ Morphism

A limit $\varprojlim F$ of the diagram is a cone (L, ϕ_B) s.t. every other cone "factors uniquely through" (L, ϕ_B)

$$\begin{tikzcd} & N \arrow[d, "\exists s"] \arrow[ddl, bend right, "\psi_i"'] \arrow[dd, bend left, "\psi_j"] \arrow[ddr, bend left, "\psi_k"] & \\ & L \arrow[dl, "\phi_i"'] \arrow[d, "\phi_j"] \arrow[dr, "\phi_k"] & \\ FB_i \arrow[r, bend left, "F(f)"] & FB_j \arrow[l, bend left] \arrow[r, bend right] & FB_k \arrow[l, bend right] \end{tikzcd}$$

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$$\begin{tikzcd} & N \arrow[d, "s"] \arrow[ddl, bend right, "\varphi_B"'] \arrow[dd, bend right, "\varphi_{B'}"'] \arrow[ddr, bend left, "\varphi_{B''}"] & \\ & L \arrow[dl, "\psi_B"'] \arrow[d, "\psi_{B'}"] \arrow[dr, dotted, "\psi_{B''}"] & \\ FB \arrow[r, "F(f)"] \arrow[loop below, "id_{F(B)}"] \arrow[rr, bend right, "F(g \circ f)"'] & FB' \arrow[r, dotted, "F(g)"] & FB'' \end{tikzcd} \qquad B, B', B'' \in \mathcal{B}$$

We can make a category of cones & a limit $\varprojlim F$ is a final (terminal) object L in this category.

Remark: Since final objects are unique up to unique isomorphism, $\varprojlim F$ is unique up to unique isomorphism, if it exists

Colimits

$$\begin{tikzcd}
FB \arrow[r, "F(f)"] \arrow[dr, "\psi_B"] \arrow[ddr, bend right, "\varphi_B"'] & FB' \arrow[r, "F(g)"] \arrow[d, "\psi_{B'}"] \arrow[dd, bend left, "\varphi_{B'}"] & FB'' \arrow[dl, "\psi_{B''}"'] \arrow[ddl, bend left, "\varphi_{B''}"] \\
 & L \arrow[d, "\exists! s"] & \\
 & N &
\end{tikzcd}$$

So $\varinjlim F = L$ is an initial object in the category of cocones

Limits	Colimits	Diagram
$\varprojlim$	$\varinjlim$	
final object	initial object	$\emptyset$
product	coproduct	$\circ^{c_1}_{id_{c_1}}$ $\circ^{c_2}_{id_{c_2}}$ $\circ$ · · · objects in $\mathcal{C}$
equilizer	coequilizer	$\circ \overset{f}{\underset{g}{\rightrightarrows}} \circ$ objects in $\mathcal{C}$
inverse limit (projective)	direct limit	Directed set
pullback	pushout	$\circ \leftarrow \circ \rightarrow \circ$ $\quad \circ \rightarrow \circ \leftarrow \circ$

Ex Recall that a directed set I $\leq$
reflexive
transitive } pre-order
(+ anti-symmetric $\Rightarrow$ poset)
(+ $a,b \in I$ have an upper bound in I $\Rightarrow$ directed set)

Ex $I = \mathbb{N} = \{1,2,3,4,5,\dots\}$ $1<2<3<\dots$
Let $\mathcal{R}$ = category of rings. Let $\mathcal{B} \subseteq \mathcal{R}$ be the category with objects, p fixed prime, $\mathbb{Z}/p^n\mathbb{Z}$, $n \geq 1$

For $i \geq 2$, $\mathbb{Z}/p^i\mathbb{Z} \xrightarrow{\varphi_i} \mathbb{Z}/p^{i-1}\mathbb{Z}$
$[n]_{p^i} \longrightarrow [n]_{p^{i-1}}$

Take F = inclusion

$$\varprojlim F = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p \text{ p-adic integers}$$

Limit

$$\begin{tikzcd}
 & & N \arrow[d, "\exists ! s"] \arrow[ddll, bend right] \arrow[ddl, bend right] \arrow[dd, bend left] \arrow[ddr, bend left] & \\
 & & L \arrow[dll] \arrow[dl, "\pi_3"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_1"] & \\
\cdots \arrow[r] & \mathbb{Z}/p^3\mathbb{Z} \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

Let's see how to find L. Embed

$$L \xhookrightarrow{\tilde{\pi}} \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$$

$$x \xmapsto{\tilde{\pi}} (\pi_1(x), \pi_2(x), \pi_3(x), \dots)$$

Claim: WLOG $\tilde{\pi}$ is 1-1

If $\tilde{\pi}$ is not 1-1. So Replace L by $L/\ker(\tilde{\pi})$

$$\begin{tikzcd}
 & & L \arrow[d, "-"] \arrow[ddll, bend right] \arrow[dd, bend left] \arrow[ddr, bend left] & \\
 & & L/\ker(\tilde{\pi}) \arrow[dll] \arrow[d] \arrow[dr] & \\
\mathbb{Z}/p^n\mathbb{Z} & \cdots & \mathbb{Z}/p^3\mathbb{Z} & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

So $L \subseteq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \cdots$

If $(a_1, a_2, a_3, \dots) \in L$ Then $\pi_2(a_2) = a_2$

$$\begin{tikzcd}
\pi_2(a_2) = a_2 \arrow[d, "\varphi_2"] \\
\pi_1(a_1) = a_1
\end{tikzcd}$$

So $L \subseteq \{(a_1, a_2, a_3, \dots),\ a_2 \equiv a_1 \pmod{p},\ a_3 \equiv a_2 \pmod{p^2}, \dots\} =: \mathbb{Z}_p$

In fact $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$

$$\begin{tikzcd}
 & R \arrow[d] \arrow[dl] \arrow[dr] & \\
\mathbb{Z}/p^n\mathbb{Z} & \mathbb{Z}_p \arrow[l, "\pi_n"'] \arrow[r, "\pi_1"] & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

$\mathbb{Z}/p^n\mathbb{Z} \quad \cdots \quad \mathbb{Z}/p\mathbb{Z}$

<u>Ex</u> Directed set $I = \mathbb{N} = \{1, 2, 3, \dots\}$, $a \leq b \Leftrightarrow a \mid b$

Let $\mathcal{C}$ = category of fields

Fix a prime p. Notice for $n \in \mathbb{N}$, $\mathbb{F}_{p^n}$:= splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$ (all roots of $\mathbb{F}_{p^n}$)

If $\mathbb{F}_{p^i} \subseteq \mathbb{F}_{p^j} \Rightarrow i \mid j$

$\mathbb{F}_{p^j} = \mathbb{F}_{p^i} 1 \oplus \cdots \oplus \mathbb{F}_{p^i} \alpha_s$

$\overset{\shortparallel}{p^j} \qquad \text{size } (p^i)^s$

Conversely if $i|j \Rightarrow \mathbb{F}_{p^i} \xhookrightarrow{\theta_{ij}} \mathbb{F}_{p^j}$ $\quad j = is$

$\alpha^{p^{is}} = \alpha$

$[\alpha + \alpha^{p^i} + \alpha^{p^{2i}} + \dots + \alpha^{i(s-1)}]$ check if is onto

What is (Colimit) $\varinjlim \mathbb{F}_{p^n}$? Category $\mathcal{B}$ objects are $\mathbb{F}_{p^i}$ $i \geq 1$
morphisms generated by $\theta_{ij} : \mathbb{F}_{p^i} \to \mathbb{F}_{p^j}$ $i|j$

$$\begin{tikzcd}
\mathbb{F}_p \arrow[r, hook] \arrow[rr, bend right] \arrow[dr] \arrow[ddr] & \mathbb{F}_{p^2} \arrow[rr, bend left] \arrow[d] & \mathbb{F}_{p^3} \arrow[dl] & \mathbb{F}_{p^4} \arrow[dll] \arrow[ddll] \\
& L \arrow[d, "\exists! s"] & & \\
& N & &
\end{tikzcd}$$

$L = \overline{\mathbb{F}_p}$ algebraic closure of $\mathbb{F}_p$.

Seen $\mathbb{Z}_p$ is a $\varprojlim$

$\overline{\mathbb{F}_p}$ is a $\varinjlim$

More generally if $(I, \leq)$ directed set

1) Category with objects $\{C_i\}_{i \in I}$ $\quad i \geq j \quad \varphi_{ij} : C_i \to C_j$

$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \quad \varphi_{ii} = id_{C_i}$

Then $\varprojlim C_i$ is the <u>inverse</u> limit of C_i

2) Category with objects $\{C_i\}_{i \in I}$ $\quad i \leq j$ -

$$\begin{tikzcd}
C_i \arrow[r, "\theta_{ij}"] \arrow[loop below, "\theta_{ii}"] \arrow[rr, bend right, "\theta_{ik}"'] & C_j \arrow[r, "\theta_{jk}"] & {}
\end{tikzcd}$$

$\varinjlim C_i$ = "directed limit" of this system

<u>Products & Coproducts</u>

Setup $\mathcal{C}$. $\mathcal{B} \subseteq \mathcal{C}$ where only morphisms are the identity
F = inclusion functor

$$\begin{tikzcd}
C \arrow[dr] \arrow[ddr] & C' \arrow[d] & C'' \arrow[dl] \arrow[ddl] \\
& L \arrow[d] & \\
& N &
\end{tikzcd}$$

coproduct $\coprod_{C\in Ob(\mathcal{B})} C$ "free product"

$$\begin{tikzcd}
& N \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& L \arrow[dl] \arrow[d] \arrow[dr] & \\
C & C' & C''
\end{tikzcd}$$

$L =$ product $\prod_{C\in Ob(\mathcal{B})} C$

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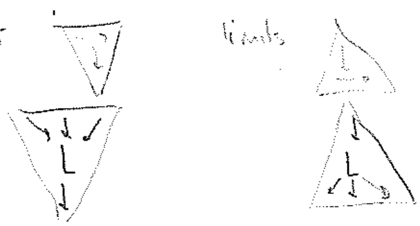
stalactites
(tomber)

stalagmites
(monter)

"Colimits are the stalactites of category theory"

(colimit
ceiling

limits



$\varinjlim$
generalizes direct limit

$\varprojlim$
generalization of inverse/projective limits

Remark: When limits/colimits exist, we can regard $\varinjlim$ or $\varprojlim$ as functors.
What does this mean? $F \ \mathcal{B} \to \mathcal{A}$
Think of all diagrams of type $\mathcal{B}$ ($\mathcal{A}$ fixed)
Identify this with $\mathrm{Funct}(\mathcal{B}, \mathcal{A})$
$F \xrightarrow{\eta} G$
Suppose that $F, G: \mathcal{B} \to \mathcal{A}$ & $\eta: F \to G$ is a natural transformation

<u>Colimit case</u>

$$\begin{tikzcd}
FB \arrow[rr, "Ff"] \arrow[dr, "\varphi_B"] \arrow[dd, "\eta_B"'] & & FB' \arrow[dl, "\varphi_{B'}"'] \arrow[dd, "\eta_{B'}"] \\
& \varinjlim F \arrow[dd, bend right, "\varinjlim \eta"] & \\
GB \arrow[rr, "Gf"] \arrow[dr, "\psi_B"'] & & GB' \arrow[dl, "\psi_{B'}"] \\
& \varinjlim G &
\end{tikzcd}$$

So $\eta : F \to G$ gives a map $\varinjlim \eta : \varinjlim F \to \varinjlim G$

$F \to \varinjlim F$
$\eta \to \varinjlim \eta$ is a functor $\qquad \varinjlim : \mathrm{Funct}(\mathcal{B}, \mathcal{A}) \to \mathcal{A}$

Overview:

1) Examples
2) Left adjoints preserve colimits, right adjoints preserve limits
3) Criteria for (small) colimits & limits to always exist

What does 2) mean?

$$D : \mathcal{D} \to \mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

Colimits

$$\begin{tikzcd} DX \arrow[rr, "Df"] \arrow[dr, "\varphi_X"'] & & DX' \arrow[dl, "\varphi_{X'}"] \\ & \varinjlim D & \end{tikzcd} \qquad D\mathcal{D}$$

Apply F. Gives a cocone

$$\begin{tikzcd} FDX \arrow[rr, "FDf"] \arrow[dr, "F\varphi_X"'] & & FDX' \arrow[dl, "F\varphi_{X'}"] \\ & F(\varinjlim D) & \end{tikzcd}$$

$\leftarrow$ colimit if F is a left adjoint

RAPL "right adjoints preserve limits"

Thm. $F(\varinjlim D) = \varinjlim FD$

	Limits	Colimits
notation	$\varprojlim$	$\varinjlim$
diagram		
Diagram $\emptyset$	final object	initial object
$\circ\ \circ\ \circ$	product	coproducts
$\circ \rightrightarrows \circ$	equalizer	coequalizer
$\circ \to \circ \leftarrow \circ$ $\quad \circ \leftarrow \circ \to \circ$	pullback	pushout
directed sets	projective limit	direct limit

Ex Coproduct in $\mathcal{G}rp$

$$\mathbb{Z}_2 = \langle x ; x^2 = 1 \rangle \qquad \mathbb{Z}_2 = \langle y ; y^2 = 1 \rangle$$

"free product" $\to \mathbb{Z}_2 \amalg \mathbb{Z}_2 \neq \langle x, y ; x^2 = y^2 = 1 \rangle \cong D_\infty$

$$x \mapsto g \qquad \mathbb{Z}_2 \amalg \mathbb{Z}_2 \to G \qquad y \mapsto h$$

$t = xy$, $\langle t \rangle \cong \mathbb{Z}$, $\langle x, y ; x^2 = y^2 = 1 \rangle = G$ Exercise: $G \cong \langle u, v ; v^2 = 1, vuv^{-1} = u \rangle$

Coproduct in $\mathcal{G}rp$, free product

$$\coprod_{i \in I} G_i = \underset{i \in I}{*} G_i \quad \text{free product}$$

Coproduct in $\mathcal{S}et$

$$\begin{tikzcd}
X \arrow[dr] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl] \arrow[ddl, bend left, "g"] \\
& X \amalg Y \arrow[d, "\exists!"] & \\
& N &
\end{tikzcd}$$

X, Y sets

Coproduct of sets is disjoint union.

Coproduct in ~~ab~~ abelian groups $A \amalg B \cong A \oplus B$ (exercise)

More generally, in R-Mod, $M \amalg N = M \oplus N$, $\coprod_{i \in I} M_i \cong \bigoplus_{i \in I} M_i$

$G : \mathcal{G}rp \to \mathcal{S}et$ forgetful functor

$F \rightleftarrows G$, $F(X)$ = free group on X

Does G have a right adjoint? No! If G is a left adjoint then it preserves colimits

Notice $G(\mathbb{Z}_2 \amalg \mathbb{Z}_2)$ is an infinite set, $G(\mathbb{Z}_2) \amalg G(\mathbb{Z}_2)$ has size 4.

So G does not preserve colimits

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Theorem: Let $D : \mathcal{D} \to \mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$. If $\varinjlim D$ exists and $\varinjlim FD$ exists then

$$F(\varinjlim D) \cong \varinjlim FD$$

ie F preserves colimits.

Proof:

$$\begin{tikzcd}
DX \arrow[rr, "Df"] \arrow[dr, "\varphi_x"] \arrow[ddr, bend right, "\beta_x"'] & & DX' \arrow[dl, "\varphi_{x'}"'] \arrow[ddl, bend left, "\beta_{x'}"] \\
& \varinjlim D & \\
& GT &
\end{tikzcd}$$

Applying F gives us a cocone

$$\begin{tikzcd}
FDX \arrow[rr, "FDf"] \arrow[dr, "F\varphi_x"] \arrow[ddr, bend right, "\alpha_x"'] & & FDX' \arrow[dl, "F\varphi_{x'}"'] \arrow[ddl, bend left, "\alpha_{x'}"] \\
& F(\varinjlim D) \arrow[d, "?"] & \\
& T &
\end{tikzcd} \quad (*)$$

Suppose we have a cocone (T, α_x).

Goal: to show $\exists!\ \theta : F(\varinjlim D) \longrightarrow T$ that makes the diagram commute.

$$\mathrm{Hom}_{\mathcal{B}}(FDX, T) \cong \mathrm{Hom}_{\mathcal{A}}(DX, GX)$$

$$\alpha_x \longleftrightarrow \beta_x$$

Claim: (GT, β_x) form a cocone.

Need to check $\beta_{x'} \circ DF = \beta_x$

Know: $\alpha_{x'} \circ FDf = \alpha_x$

$A \xrightarrow{\varphi} A'$ $\quad B \xrightarrow{\psi} B'$

$$\begin{tikzcd}
f \in \mathrm{Hom}(A', GB) \arrow[d, mapsto] \arrow[r] & \mathrm{Hom}(FA', B) \\
G(\psi)\circ f\circ\varphi \in \mathrm{Hom}(A, GB') \arrow[r] & \mathrm{Hom}(FA, B')
\end{tikzcd}$$

Guess: $A = DX \xrightarrow{Df} A' = DX'$

$B = B' = T \quad T \xrightarrow{id_T} T$

$$\begin{tikzcd}
\beta_{x'} \arrow[r, leftrightarrow] & \alpha_{x'} \\
\mathrm{Hom}(DX', GT) \arrow[r] \arrow[d] & \mathrm{Hom}(FDX', T) \arrow[d] \\
\mathrm{Hom}(DX, GT) \arrow[r] & \mathrm{Hom}(FDX, T) \\
\beta_{x'}\circ Df = \beta_x \arrow[r, leftrightarrow] & \alpha_x = \alpha_{x'}\circ FDf
\end{tikzcd}$$

So $\exists!\ u : \varinjlim D \to GT$

$$\mathrm{Hom}(\varinjlim D, GT) \cong \mathrm{Hom}(F(\varinjlim D), T)$$
$$u \longleftrightarrow \theta$$

Claim: θ makes the diagram (*) commute & is the unique morphism with this property. Ie we must show $\alpha_x = \theta\circ F\varphi_x \quad \forall X\in Ob(\mathcal{B})$

know $\beta_x = u\circ\varphi_x$

$$\begin{tikzcd}
u \arrow[r] & \theta \\
\mathrm{Hom}(\varinjlim D, GT) \arrow[r] \arrow[d] & \mathrm{Hom}(F(\varinjlim D), T) \arrow[d] \\
\mathrm{Hom}(DX, GT) \arrow[r] & \mathrm{Hom}(FDX, T) \\
u\circ\varphi_x = \beta_x \arrow[r, bend right] & \alpha_x = \theta\circ F\varphi_x
\end{tikzcd}$$

So $\alpha_x = \theta\circ F\varphi_x \implies \theta$ makes the diagram commute. Easy to see: θ is unique ∎

Corollary: If A is a commutative ring & M is an A-module

$-\otimes_A M : A\text{-Mod} \to A\text{-Mod}$

$N \longmapsto N\otimes_A M$

$N \xrightarrow{f} N' \longrightarrow N\otimes_A M \xrightarrow{f\otimes id} N'\otimes_A M$

$n\otimes m \longmapsto f(n)\otimes m$

Then

1) $-\otimes_A M$ is right-exact ie if $N' \xrightarrow{f} N \xrightarrow{h} N'' \to 0$ is exact then
$N'\otimes M \xrightarrow{f\otimes id} N\otimes M \xrightarrow{h\otimes id} N''\otimes M \to 0$ is exact

2) $\left(\bigoplus_{i\in I} N_i\right)\otimes_A M \cong \bigoplus_{i\in I}(N_i\otimes_A M)$

Proof?

2) Recall coproduct of modules in A-Mod $\{N_i\}_{i\in I}$

$$\begin{tikzcd}
N_i \arrow[dr, "u_i"'] \arrow[ddr, bend right, "f_i"'] & N_j \arrow[d, "u_j"] \arrow[dd, bend left, "f_j"] & N_k \arrow[dl, "u_k"] \arrow[ddl, bend left, "f_k"] \\
 & \bigoplus_{i\in I} N_i \arrow[d, dashed, "\exists"] & \\
 & L &
\end{tikzcd}$$

$(n_i)_{i\in I} \longmapsto \sum_i f_i(n_i)$

$\varinjlim N_i = \bigoplus N_i$

$$\left(\varinjlim N_i\right)\otimes_A M \cong \varinjlim N_i \otimes_A M$$
$$\left(\bigoplus_i N\right)\otimes_A M \cong \bigoplus_i (N_i \otimes_A M)$$

<u>Right exactness</u>

$$N' \xrightarrow{f} N \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$
$$\operatorname{coker}(f) = N/f(N')$$

Notice that a cokernel = coequalizer & kernel = equalizer

Remark: In A-Mod

$$\begin{tikzcd}
 & \ker(f-g) \arrow[dl] \arrow[dr] & \\
N' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & N \arrow[dl] \\
 & N/\operatorname{Im}(f-g) &
\end{tikzcd}$$

equalizer $\ker(f-g)$, coequalizer $N/\operatorname{Im}(f-g)$

In particular,

$$\begin{tikzcd}
 & \operatorname{kernel}(f) \arrow[dl] \arrow[dr] & \\
N' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "0"'] \arrow[dr] & & N \arrow[dl] \\
 & \operatorname{cokernel}(f) &
\end{tikzcd}$$

Why?

$$\begin{tikzcd}
 & L \arrow[d, "\exists! u"] \arrow[dl, "u"'] \arrow[dr, "v"] & \\
N' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & \ker(f-g) \arrow[l] \arrow[r] & N
\end{tikzcd}$$

$u(x)$ $\quad$ $v(x)$

$f(u(x)) = g(u(x)) = v(x) \;\forall x\in L$
$\longrightarrow u(x)\in\ker(f-g)$

For colimits it is similar

$$\begin{tikzcd}
N' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "0"'] \arrow[dr] & & N \arrow[dl] \\
 & \operatorname{coker}(f) &
\end{tikzcd}$$

& since $-\otimes_A M$ preserves colimits,

$$\begin{tikzcd}
N'\otimes M \arrow[rr, shift left, "f\otimes id"] \arrow[rr, shift right, "0"'] \arrow[dr] & & N\otimes M \arrow[dl] \\
 & \operatorname{coker}(f\otimes id) \cong \operatorname{coker}(f)\otimes_A M &
\end{tikzcd}$$

In other words,

$$N'\otimes M \xrightarrow{f} N\otimes M \to (N/\mathrm{Im}(f))\otimes_A M \text{ is exact}$$

So $-\otimes_A M$ is a right exact functor.

Dually, $\otimes$-Hom adjunction Gives $\mathrm{Hom}(M,-)$ is left exact

$$0 \to A = \ker(f) \hookrightarrow B \xrightarrow{f} C$$

Then $0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C)$ is exact

$$\psi \mapsto f\circ\psi$$

products

$\prod M_i$ is ordinary Cartesian product

$$\begin{tikzcd}
 & L \arrow[d] \arrow[ddl, bend right, "f_i"'] \arrow[dd, bend left, "f_j"] & \\
 & \prod M_i \arrow[dl, "\pi_i"'] \arrow[d, "\pi_j"] & \\
M_i & M_j &
\end{tikzcd}$$

product check

$$\mathrm{Hom}_A(M,\prod N_i) \cong \prod \mathrm{Hom}_A(M,N_i)$$

$$(f_i)_{i\in I}: M \to \prod N_i$$

Tensor product is not left exact!

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}: \quad \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Def) An A-module M is flat if $0 \to N' \xrightarrow{f} N$ exact $\Rightarrow 0 \to N'\otimes M \xrightarrow{f\otimes id} N\otimes M$ is exact

Ex $M = A^I$, free module, is flat

If $N' \xrightarrow{f} N$ injective

$$\begin{array}{ccc}
N'\otimes A^I & \longrightarrow & N\otimes A^I \\
\| \wr & & \\
\bigoplus_{i\in I}(N'\otimes_A A) & \xrightarrow{(f\otimes id)_{i\in I}} & \bigoplus_i N\otimes_A A \\
\| \wr & & \\
\bigoplus_{i\in I} N' & \xrightarrow{(f)_{i\in I}} & \bigoplus_{i\in I} N
\end{array}$$

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Criterion for existence of small colimits

Def) Such a category is called cocomplete (complete for small limits)

complete + cocomplete = bicomplete

Theorem: Let $\mathcal{C}$ be a category & suppose all small coproducts exist & all coequalizers $C \underset{g}{\overset{f}{\rightrightarrows}} C'$ exist $\Rightarrow$ $\mathcal{C}$ is cocomplete

Proof: ~~Say~~ Let $F : \mathbb{B} \to \mathbb{C}$ be a small diagram. Goal: Show $\varinjlim F$ exists.
Let $C' = \coprod_{B \in Ob(\mathbb{B})} FB \in Ob(\mathbb{C})$

$$\begin{tikzcd} FB \arrow[dr, "i_B"'] & FB' \arrow[d, "i_{B'}"] & FB'' \arrow[dl, "i_{B''}"] \\ & C' & \end{tikzcd}$$

$F\mathbb{B}$

Remark: Let $Mor(\mathbb{B})$ = set of morphisms $\varphi : B \to B'$ in $\mathbb{B}$, $B, B' \in Ob(\mathbb{B})$
Notice each $\varphi \in Mor(\mathbb{B})$ has a source and a target.

$$\varphi : \underset{\text{source} = S(\varphi)}{B} \to \underset{t(\varphi) = \text{target}}{B'}$$

Let
$$C = \coprod_{\varphi \in Mor} F(S(\varphi)) \in Ob(\mathbb{C})$$

$$B \xrightarrow{\varphi} B'$$

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] & FB' \arrow[d, "\alpha_{B'}"] \\ & C \end{tikzcd}$$

φ, $id_{B'}$

cocone coming from C as coproduct

Now we construct morphisms $\Phi, \Psi : C \to C'$ & we'll show $\varinjlim F$ = coequalizer (Φ, Ψ)

$\varphi' : B \to B'$ $\quad B \xrightarrow{\varphi} B'$

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] \arrow[ddr, bend right, "i_B"'] & & FB' \arrow[dl, "\alpha_{B'}"'] \arrow[ddl, bend left, "i_{B'}"] \\ & C \arrow[d, "\exists!\,\Phi"] & \\ & C' & \end{tikzcd}$$

φ

$i_B = \Phi \circ \alpha_B$

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] \arrow[ddr, bend right, "i_{B'} \circ F(\varphi)"'] & & FB' \arrow[dl, "\alpha_{B'}"'] \arrow[ddl, bend left, "i_{t(\tilde\varphi)} \circ F(\tilde\varphi)"] \\ & C \arrow[d, "\exists!\,\Psi"] & \\ & C' & \end{tikzcd}$$

φ $\quad \tilde\varphi$

$$F(\varphi) : FB \to FB'$$
$$\downarrow i_{B'}$$
$$C'$$

$i_{B'} \circ F(\varphi) = \Psi \circ \alpha_B \quad \varphi : B \to B'$
$i_{t(\varphi)} \circ F(\varphi) = \Psi \circ \alpha_{S(\varphi)}$

By assumption, coequalizers exist

$$\begin{tikzcd} C \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] \arrow[dr, "u"'] & C' \arrow[d, "v"] \\ & L \end{tikzcd}$$

L = coequalizer

$u = v \circ \Phi$
$u = v \circ \Psi$

Claim: $L \cong \varinjlim F$. Let's see why!

$$\begin{tikzcd}
FB \arrow[r, "F\varphi"] \arrow[dr, "\gamma_B"] \arrow[ddr, bend right, "\theta_B"'] & FB' \cdots \arrow[d, "\gamma_{B'}"'] \arrow[dd, bend left, "\theta_{B'}"] \\
 & L \arrow[d, "s"] \\
 & T
\end{tikzcd}$$

Suppose we have a cocone (T, θ_B). Claim: $\exists$ morphisms $\gamma_B : F_B \to L$ & $\exists!\ s : L \to T$ st the diagram commutes

$$\begin{tikzcd}
FB \arrow[r, "F\varphi"] \arrow[dr, "i_B"] \arrow[ddr, bend right, "\gamma_B"'] & FB' \cdots \arrow[d, "i_{B'}"] \\
 & C' \arrow[d, "v"] \\
 & L
\end{tikzcd}$$

1) Need to check that (L, γ_B) is a cocone, i.e. $\gamma_B = \gamma_{B'} \circ F\varphi$ $\forall \varphi : B \to B'$.

$\gamma_B = v \circ i_B$ $\quad \gamma_{B'} = v \circ i_{B'}$

Must show $v \circ i_B = v \circ i_{B'} \circ F\varphi$

$$\begin{array}{cc}
\shortparallel & \shortparallel \\
v \circ \Phi \circ \alpha_B & v \circ \psi \circ \alpha_B \\
\shortparallel & \shortparallel \\
u \circ \alpha_B & u \circ \alpha_B \ \checkmark
\end{array}$$

Next, suppose that (T, θ_B) is a cocone

$$\begin{tikzcd}
FB \arrow[r, "F\varphi"] \arrow[dr, "i_B"] \arrow[ddr, bend right, "\theta_B"'] & FB' \arrow[d, "i_{B'}"'] \arrow[dd, bend left, "\theta_{B'}"] & \\
 & C \arrow[r, "v"] \arrow[d, "\exists! h"] & L \arrow[dl] \\
 & T &
\end{tikzcd}$$

We don't know (C, i_B) is a cocone

So $\exists!\ h : C \to T$ $\quad h \circ i_B = \theta_B$

Now

$$\begin{tikzcd}
FB \arrow[dr] \arrow[ddr, bend right] & FB' \arrow[d] & \\
 & C' \arrow[r, "v"] \arrow[d, "h"] & L \arrow[dl, "\exists! \hat{h}"] \\
 & T &
\end{tikzcd}$$

To get h to factor through L we must show

$$\begin{tikzcd}
C \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] \arrow[dr] & C' \arrow[d, "h"] \\
 & T
\end{tikzcd}$$

$h \circ \Phi = h \circ \Psi$

Must show $h \circ \Phi = h \circ \Psi$

$$
\begin{array}{c|c}
h\circ\Phi\circ\alpha_B & h\circ\Psi\circ\alpha_B \\
\shortparallel & \shortparallel \\
h\circ i_B & h\circ i_{B'}\circ F(\varphi) \\
\shortparallel & \shortparallel \\
\theta_B & = \theta_{B'}\circ F(\varphi)
\end{array}
$$

$$
\begin{tikzcd}
FB \arrow[dr, "\alpha_B"] \arrow[ddr, bend right, "\theta_B"'] & & FB' \arrow[dl, "\alpha_{B'}"'] \arrow[ddl, bend left, "\theta_{B'}"] \\
& C \arrow[d, bend right, "h\circ\Phi"'] \arrow[d, bend left, "h\circ\Psi"] & \\
& T &
\end{tikzcd}
$$

By uniqueness, $\because C$ is the coproduct & both $h\circ\Phi$ & $h\circ\Psi$ ~~work~~ make the diagram commute $\Rightarrow h\circ\Phi = h\circ\Psi$ ∎

Remark: the exact same argument shows that if $F: \mathcal{C}\to\mathcal{D}$ & suppose $\mathcal{C}, \mathcal{D}$ are cocomplete &

$$F\Big(\coprod_{i\in I} C_i\Big) \cong \coprod_{i\in I} FC_i$$

$$F\big(\mathrm{Coequal}(C \overset{f}{\underset{g}{\rightrightarrows}} C')\big) \cong \mathrm{Coequal}\,(FC \overset{Ff}{\underset{Fg}{\rightrightarrows}} FC')$$

Then $F(\varinjlim D) \cong \varinjlim FD$ $\forall$ small diagrams $D: \mathcal{B}\to\mathcal{C}$

Cor: The following categories are bicomplete

Category	Product	Coproduct	Equal	Coeq.
Abelian groups	$\prod A_i$	$\oplus A_i$	$\ker(f-g)$	$\mathrm{coker}(f-g)$
R-Mod	$\prod M_r$	$\oplus M_r$	$\ker(f-g)$	$\mathrm{coker}(f-g)$
Comm. rings	$\prod R_i$	$\bigotimes_{\mathbb{Z}} R_i$	$\{f(x)=g(x)\}$	$R/\langle f(x)-g(x)\rangle$
groups				