"I can see the appeal of talking to a turtle."

"Why do I want to talk to a bat? I don't know."

"If I scream enough times in different locations."

Yoneda's Lemma

Ex $\mathcal{Ab}_{fin}$ = finite abelian groups

$A \in Ob(\mathcal{Ab}_{fin})$ for all finite abelian groups B

$$|Hom_{Ab}(A,B)|$$

Q: Can I recover A if I know $|Hom(A,B)|$ $\forall$ finite abelian groups B?

Q: Equivalently, if $A_1 \ncong A_2 \Rightarrow \exists B$ st $|Hom(A_1,B)| \neq |Hom(A_2,B)|$

Ex $A_1 = \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_5$

$A_2 = \mathbb{Z}_2^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5$

$|Hom(A_1, \mathbb{Z}_5)| = 5$

$|Hom(A_2, \mathbb{Z}_5)| = 5$

$|Hom(A_1, \mathbb{Z}_2)| = 2^6$

$|Hom(A_2, \mathbb{Z}_2)| = 2^6$

$|Hom(A_1, \mathbb{Z}_4)| = 2^3 \cdot 4^3$

$|Hom(A_2, \mathbb{Z}_4)| = 2^4 \cdot 4 \cdot \overset{\#}{4}$

Yoneda's lemma says roughly that we can understand objects $A \in Ob(\mathcal{A})$ by understanding $Hom_{\mathcal{A}}(A,B)$ $\forall B \in Ob(\mathcal{A})$.

Representable functors

Let $\mathcal{A}$ be a category. Let $A \in Ob(\mathcal{A})$. Then we can make a functor

$$h_A : \mathcal{A} \to \mathcal{S}et$$

For $B \in Ob(\mathcal{A})$, $h_A(B) := Hom_{\mathcal{A}}(A,B)$ & if $B \xrightarrow{f} B'$

$$\begin{array}{ccc} h_A(B) & \xrightarrow{h_A(f)} & h_A(B') \\ \parallel & & \parallel \\ Hom(A,B) & & Hom(A,B') \\ \psi & \xmapsto{h_A(f)} & f \circ \psi \end{array}$$

So h_A is called a representable functor.

On the assignment

$Funct(\mathcal{A}, \mathcal{S}et)$

Objects are functors from $\mathcal{A} \to \mathcal{S}et$

morphisms $\eta : F \to G$ natural transformations

Let $\mathcal{F}$ be the (full) subcategory of Fund$(\mathcal{A}, \mathcal{S}et)$ whose objects are representable functors &

$\mathrm{Hom}_{\mathcal{F}}(h_A, h_B) = \{$natural transformations from h_A to $h_B\}$

Theorem [Yoneda's Lemma]:

$$\mathcal{A} \cong \mathcal{F}^{op}$$

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Example

$\mathcal{G}rp$ = category of groups

$G : \mathcal{G}rp \to \mathcal{S}et$ $\quad G(H) = H$ forgetful functor

Then G is a representable functor.

We need $A \in Ob(\mathcal{G})$, $\mathrm{Hom}_{\mathcal{G}rp}(A, H) \cong H$

Take $A = \mathbb{Z}$

$G \cong h_A$

$F : \mathcal{S}et \to \mathcal{G}rp$ $\quad F(X)$ = free group on X. Then $\mathbb{Z} = F(\{x\})$

$$\mathrm{Hom}_{\mathcal{G}rp}(F(X), H) \cong \mathrm{Hom}_{\mathcal{S}et}(X, GH)$$

So $= \mathrm{Hom}(\{x\}, H) \cong \mathrm{Hom}_{\mathcal{G}rp}(F(\{x\}), H) = \mathrm{Hom}_{\mathcal{G}rp}(\mathbb{Z}, H)$

k = field, $\mathcal{C}$ = commutative k-algebras

$k \xhookrightarrow{i} C \quad 1_k \longmapsto 1_c$

Given a $C \in Ob(\mathcal{C})$ We can form a category C-Mod

If M is a C-module. A <u>derivation</u> $\delta : C \to M$ is a k-linear map

$\delta(c_1 c_2) = c_1 \delta(c_2) + c_2 \delta(c_1)$

$\mathrm{Der}_k(C, M)$ = all k-linear $\delta : C \to M$

This is a C-module $\quad (c \cdot \delta)(a) := c\,\delta(a)$

So we have a functor $\mathrm{Der} : C\text{-Mod} \to C\text{-Mod}$

$$M \longmapsto \mathrm{Der}_k(C, M)$$

$M_1 \xrightarrow{f} M_2 \rightsquigarrow \mathrm{Der}_k(C, M_1) \xrightarrow{\mathrm{Der}(f)} \mathrm{Der}_k(C, M_2)$

$$\delta \longmapsto f \circ \delta$$

$(f \circ \delta)(ab) = \cdots = a f(\delta(b)) + b f(\delta(a))$

Claim: Der is representable

Kähler differentials, C k-algebra, C-module $\Omega_{C/k}$
Free C-module on all symbols dc, $c \in C$ modulo the relations
$$d(c_1 + \lambda c_2) - dc_1 - \lambda dc_2 = 0, \quad d(c_1 c_2) = c_1 dc_2 + c_2 dc_1, \quad d\lambda = 0$$
The quotient is $\Omega_{C/k}$
$\mathrm{Der} \cong h_{\Omega_{C/k}}$ ie
$$\begin{array}{c} \mathrm{Der}(M) = \mathrm{Der}_k(C, M) \\ \cong \\ \mathrm{Hom}_C(\Omega_{C/k}, M) \end{array}$$

Ex $C = k[t]$, $\Omega_{k[t]/k}$, $dp(t)$
$$d(a_0 + a_1 t + \cdots + a_s t^s) = a_0 d1 + a_1 dt_1 + \cdots + a_s dt^s$$
$$= \underbrace{a_0 dt}_{0} + a_1 dt + 2a_2 t\, dt + \cdots + s a_s t^{s-1} dt$$

$$\Omega_{k[t]/k} = k[t]\,dt$$
$$\begin{array}{c} \mathrm{Der}_k(k[t], M) \\ \cong \\ \mathrm{Hom}_{k[t]}(\Omega_{k[t]/k}, M) \end{array}$$

$$\delta : k[t] \longrightarrow \underset{}{\overset{k[t]\text{-module}}{M}}$$
$$\updownarrow$$
$$f_\delta : \Omega_{k[t]/k} \longrightarrow M$$
$$f_\delta(dt) = \delta(t)$$
In general $f_\delta(dc) = \delta(c)$
$$f_\delta(p(t)dt) = p(t)\delta(t)$$
Conversely, if
$$f : \Omega_{k[t]/k} \longrightarrow M$$
$$\updownarrow$$
$$\delta_f : k[t] \longrightarrow M$$
$$\delta_f(p(t)) = f(dp(t)) = f(p'(t)dt) = p'(t)f(dt)$$
$$\delta_f(p(t)) = p'(t) f(dt)$$
$$\delta_f(p(t)q(t)) = (p(t)q(t))' * f(dt) = \cdots = q(\delta_f(p)) + p(\delta_f(q))$$

(Topological) presheaves
$$\updownarrow$$
Yoneda's lemma

$Top_X \leftarrow$ objects $U \subseteq X$ open
X topological space

$$\mathrm{Hom}_{Top_X}(U,V) = \begin{cases} i & \text{if } U \overset{i}{\hookrightarrow} V \\ \emptyset & \text{otherwise} \end{cases}$$

Top_X^{op} a presheaf (of sets) is a functor

$$S : Top_X^{op} \longrightarrow \mathcal{S}et$$

$$S : Top_X \longrightarrow \mathcal{S}et \quad \text{controvariant}$$

$$U \overset{i}{\hookrightarrow} V \rightsquigarrow S(V) \xrightarrow[\underset{P_{V,U}}{\shortparallel}]{S(i)} S(U)$$

Think of p_{UV} as "restriction" from V to U

$$f \in S(V) \xrightarrow{S(i)} f|_U$$

Ex X = top space

$$\mathcal{O} : Top_X^{op} \longrightarrow \mathcal{S}et$$

$$U \longmapsto \{f : U \to \mathbb{C} \text{ cts}\}$$

$$U \overset{i}{\hookrightarrow} V$$

$$f \in \mathcal{O}(V) \xrightarrow{S(i)} f|_U \in \mathcal{O}(U)$$

What are the representable functors $h : Top_X^{op} \to \mathcal{S}et$?

$U \subseteq X$ open $\quad h_U : Top_X^{op} \to \mathcal{S}et$

$$h_U(V) = \mathrm{Hom}_{Top_X^{op}}(U,V) = \mathrm{Hom}_{Top_X}(V,U)$$

$h_U(V)$ is empty if $V \not\subseteq U$
$h_U(V)$ $\{i : V \to U\}$ otherwise

If h_U & S are two presheaves

$$h_U : Top_X^{op} \to \mathcal{S}et$$

$$S : Top_X^{op} \to \mathcal{S}et$$

What is a natural transformation $\eta : h_U \to S$?

$\forall V \subseteq X$ open

$V \xrightarrow{Top_X^{op}} V'$

$V' \hookrightarrow V$

$$\begin{array}{ccc} h_U(V) & \xrightarrow{\eta_V} & S(V) \\ \downarrow & & \\ h_U(V') & \xrightarrow{\eta_{V'}} & S(V') \end{array}$$

Claim: η is completely determined by η_U

If $V \hookrightarrow U$

$$\begin{array}{ccc} h_U(U) & \xrightarrow{\eta_U} & S(U) \\ \downarrow & & \downarrow \\ h_U(V) & \xrightarrow{\eta_V} & S(V) \end{array}$$

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Ex Topological presheaves

X = top. space

Top_X Objects $U \subseteq X$ open, $U \xhookrightarrow{i} V$ morphisms

A presheaf of $\mathcal{X}$ ($\mathcal{X}$ = sets, rings, groups, abelian groups, etc)

Is just a contravariant functor $\mathcal{F}: Top_X \to \mathcal{X}$ where $Ob(\mathcal{X}) = \mathcal{X}$

Ex Let $X = \mathbb{C}$ with Euclidean topology

$\mathcal{F}: Top_X^{op} \to Ring$

$\mathcal{F}(U) = \{f: U \to \mathbb{C};\ f \text{ is analytic on } U\}$

If $U \subseteq V$, $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

$f \longmapsto f|_U$

A presheaf $\mathcal{F}: Top_X^{op} \to \mathcal{C}$ is a sheaf if it has two additional conditions

1) It is separated, i.e. if $U \subseteq X$ open & $U = \bigcup_{i \in I} U_i$

Then if $f, g \in \mathcal{F}(U)$ are such that $f|_{U_i} = g|_{U_i}\ \forall i \Rightarrow f = g$

$$\mathcal{F}(U) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(U_i)$$

$$f \longmapsto f|_{U_i} = \mathcal{F}(i)(f)$$

2) Glue $U = \bigcup_{i \in I} U_i$. If $\exists f_i \in \mathcal{F}(U_i)$ $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}\ \forall i, j$

$\Rightarrow \exists f \in \mathcal{F}(U)$ st $f|_{U_i} = f_i\ \forall i$

Eg. $\mathcal{F}: Top_X^{op} \to Ring$ $\mathcal{F}(U) = \{f: U \to \mathbb{C};\ f \text{ is cts}\}$. This is a sheaf

Ex

$X = \mathbb{R}$, Euclidean topology

$\mathcal{F}(U) = \{$bdd cts fncs from $U \to \mathbb{R}\}$

$$V \subseteq U$$
$$f \in \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$
$$f \longmapsto f|_V$$

Don't have gluing:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} \underbrace{(-n, n)}_{U_n}$$

$f_n(x) = x \in \mathcal{O}(U_n)$ & $f_n|_{U_n \cap U_m} = f_m|_{U_n \cap U_m}$
But $\nexists f: \mathbb{R} \to \mathbb{R}$ bded st $f|_{U_n} = f_n \ \forall n$

$G: Shv \to PreShf$

$\mathcal{C} = Set$
Representable presheaf $U \subseteq X$ open

$$h_U : Top_X^{op} \to Set \quad h_U(V) = Hom(U,V)$$

$$V_1 \xhookrightarrow{i} V_2 \quad \text{in } Top_X$$

$$V_2 \longrightarrow V_1 \quad \text{in } Top_X^{op}$$

$$h_U(V_2) \to h_U(V_1) \text{ in } Set$$

$$\begin{array}{ccc} \| & & \\ Hom_{Top_X^{op}}(U,V_2) & \to & Hom_{Top_X^{op}}(U,V_1) \\ \| & & \\ Hom_{Top_X}(V_2,U) & \longrightarrow & Hom_{Top_X}(V_1,U) \\ \psi & \longmapsto & \psi \circ i \end{array}$$

What is a natural transformation

$$\eta : h_U \to \mathcal{F} ?$$

$$h_U(V) \xrightarrow{\eta_V} \mathcal{F}(V)$$

$V_2 \to V_1$ in Top_X^{op} (i.e. $V_1 \xhookrightarrow{i} V_2$)

$$\begin{tikzcd} h_U(V_2) \arrow[r, "\eta_{V_2}"] \arrow[d, "h_U(i)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(V_2) \arrow[d, "\mathcal{F}(i)"] \\ h_U(V_1) \arrow[r, "\eta_{V_1}"] & \mathcal{F}(V_1) \end{tikzcd}$$

Claim: $\eta : h_U \to \mathcal{F}$ η is completely determined by $\eta_U : h_U(U) \to \mathcal{F}(U)$

$$id_U \longrightarrow \eta_U(id_U)$$

What if $\mathcal{F} = h_V$? Then $\eta : h_U \to h_V$

$$\Big\{ \text{bij}$$

$$h_V(U) = Hom_{Top_X^{op}}(V,U) = Hom_{Top_X}(U,V)$$

Why do we get the claim?
Case I: $V \subseteq_{open} U$ $V \xhookrightarrow{i} U$ $U \to V$ in Top_X^{op}

$$\begin{tikzcd} id_U & \\ h_U(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ h_U(V) \arrow[r, "\eta_V"] & \mathcal{F}(V) \end{tikzcd}$$

$$\overset{\|}{Hom(U,V)}$$

So η_V is determined by η_U if $V \subseteq U$ open

Case II: $V \not\subseteq U \Rightarrow h_U(V) = \emptyset$

Proof (of Yoneda's Lemma):

Category $\mathcal{A}$

Objects $A, B, C, \dots$

Morphisms $A \xrightarrow{f} B$

Category $\mathcal{F} \subseteq \mathcal{F}unct(\mathcal{A}, \mathcal{S}et)$

objects $h_A, h_B, h_C, \dots$

Morphisms $\eta : h_A \to h_B$

Claim: $\mathcal{A} \cong \mathcal{F}^{op}$

Need to construct $F : \mathcal{A} \to \mathcal{F}^{op}$, $G : \mathcal{F}^{op} \to \mathcal{A}$

Define $F(A) = h_A$

$$A \xrightarrow{f} B$$
$$h_B \xrightarrow{\eta_f = F(f)} h_A$$

$\forall C \in Ob(\mathcal{A})$, $h_B(C) \xrightarrow{(\eta_f)_C} h_A(C)$

$$\begin{aligned} \mathrm{Hom}(B,C) &\longrightarrow \mathrm{Hom}(A,C) \\ \psi &\longmapsto \psi \circ f \end{aligned}$$

To check that $\eta_f : h_B \to h_A$ is a natural transformation, let $g : C \to C'$, $C, C' \in Ob(\mathcal{A})$

$$\begin{tikzcd}
\psi \arrow[rrr, mapsto] \arrow[d, mapsto] & h_B(C) \arrow[r, "(\eta_f)_C"] \arrow[d] & h_A(C) = \mathrm{Hom}(A,C) \arrow[d] & \psi \circ f \arrow[d, mapsto] \\
g \circ \psi \arrow[rrr, mapsto] & h_B(C') \arrow[r, "(\eta_f)_{C'}"] & h_A(C') = \mathrm{Hom}(A,C') & g \circ \psi \circ f
\end{tikzcd}$$

So the map $A \to h_A$, $f \mapsto \eta_f$ is a functor $F : \mathcal{A} \to \mathcal{F}^{op}$.

Let $G : \mathcal{F}^{op} \to \mathcal{A}$ $\quad G(h_A) = A$

$\eta : h_A \longrightarrow h_B \sim \underset{G(\eta)}{\overset{\shortparallel}{f(\eta)}} : B \to A$

$\eta_A : h_A(A) \longrightarrow h_B(A)$

$\mathrm{Hom}(A,A) \longrightarrow \mathrm{Hom}(B,A)$

Define $G(\eta) =: f(\eta)$

$\qquad \overset{\shortparallel}{\eta_A(id_A)} : B \to A$

Check G is a functor.

Look at $G\circ F : \mathcal{A} \to \mathcal{A}$, $F \circ G : \mathcal{F}^{op} \to \mathcal{F}^{op}$

Claim: these are the identity functors

$$G\circ F(A) = G(h_A) = A \checkmark$$
$$F\circ G(A) = F(A) = h_A \checkmark$$

$$A \xrightarrow{f} B$$
$$A \xrightarrow{GF(f)} B$$

We need to check $GF(f) = f$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, squiggly] & B \\
h_B \arrow[r, "\eta_f"] \arrow[d, squiggly] & h_A \\
A \arrow[r, "f(\eta_f)"] & B
\end{tikzcd}$$

$$f(\eta_f) = (\eta_f)_B(id_B)$$

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First $G\circ F(f) = G(\eta_f) = (\eta_f)_B(id_B) = id_B \circ f = f \checkmark$

Next $F\circ G(\eta) = F(f(\eta)) = F(\eta_B(id_B)) \overset{?}{=} \eta$

Need to show $F(\underbrace{\eta_B(id_B)}_{g})_C = \eta_C \ \forall C \in Ob(\mathcal{A})$

So $F(g) = \eta_g$, $(\eta_g)_C = - \circ g$ So we must show

$$\eta_C = - \circ (\eta_B(id_B))$$

$$\begin{tikzcd}
id_B \arrow[rrr] \arrow[d] & h_B(B) \arrow[r, "\eta_B"] \arrow[d] & h_A(B) \arrow[d] & g \arrow[d] \\
\psi \arrow[rrr, bend right] & h_B(C) \arrow[r, "\eta_C"] & h_A(C) & \psi\circ g
\end{tikzcd}$$

$\psi \in h_B(C)$, $\psi : B \to C$

$\Rightarrow \eta_C(\psi) = \psi \circ g \checkmark$

G category

$$h_C(C) = Hom(C,C) \xrightarrow{f^*} Hom(C,C)$$

Corollary: Any small category is "concretizable", i.e. the category is equivalent to a category in which each object is a set.

Idea: Let $\mathcal{C}$ be your small category

$$\mathcal{C} \overset{\text{Yoneda}}{\cong} \mathcal{F}^{op} \subseteq \text{Funct}(\mathcal{C}, \mathcal{S}et)^{op}$$

$$C \longmapsto h_C$$

$h_C(B) = \text{Hom}_{\mathcal{C}}(C, B) \leftarrow$ set

We make a new category $\tilde{\mathcal{C}}$ objects are given as follows

$\forall B \in Ob(\mathcal{C})$ Make a set

$$\hat{B} = \coprod_{C \in Ob(\mathcal{C})} h_B(C)$$

($h_B(C)$ ← set; $C \in Ob(\mathcal{C})$ ← set)

$f : B \to B'$

$$\hat{B'} \xrightarrow{\hat{f}} \hat{B}$$

$$\coprod_C h_{B'}(C) \xrightarrow{\hat{f}} \coprod_C h_B(C)$$

$$\varphi_C \longmapsto \varphi_C \circ f$$

$$\underset{h_{B'}(C)}{\overset{\cap}{}} \longmapsto \varphi_C \circ f$$

This gives us a concrete category $\tilde{\mathcal{C}}$ & $\mathcal{C} \cong \mathcal{F}^{op} \cong \tilde{\mathcal{C}}^{op}$