2016 04 01

Group Cohomology

G-Mod (G group, fixed) is the category of abelian groups endowed with a G-action $G \times A \to A$ $(A = (A, +))$

$H: G\text{-Mod} \to Ab$

$$A \mapsto HA = \{a \in A \,;\, ga = a \ \forall g \in G\}$$

Ex. $G = S_2$, $A = \mathbb{Z} \oplus \mathbb{Z}$, $(12)(a,b) = (b,a)$

$$HA = \{(a,b) \,;\, id(a,b) = (a,b),\ (12)(a,b) = (a,b)\}$$
$$\parallel$$
$$\mathbb{Z}(1,1) \cong \mathbb{Z}$$

Remark: H is left exact

Why? If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is exact $A, B, C \in G\text{-Mod}$

then $0 \to HA \xrightarrow{H\varphi} HB \xrightarrow{H\psi} HC$ is exact.

Remark: H is not right-exact

eg. $G = \mathbb{Z}/2\mathbb{Z} = \langle x \mid x^2 = 1 \rangle$

$B = \mathbb{Z}/4\mathbb{Z}$, $C = \mathbb{Z}/2\mathbb{Z}$

$B \to C$ $\qquad G \circlearrowright B \qquad G \circlearrowright C$

$0, 2 \longmapsto 0 \qquad x \cdot 1 = 3 \qquad x \cdot 1 = 1 \qquad HB \to HC$ is zero map

$1, 3 \longmapsto 1$

Remark: H commutes with proj. limits (exercise):

$$H(\varprojlim M_i) \cong \varprojlim HM_i$$

Remark: $G\text{-Mod} \cong \mathbb{Z}[G]\text{-Mod}$

$$\mathbb{Z}[G] = \{\textstyle\sum_{g \in G} n_g g \,;\, n_g = 0 \text{ for all but finitely many } n_g \in \mathbb{Z}\}$$

$(n \cdot g)(m \cdot h) = nm \cdot gh$

So view $H: \mathbb{Z}[G]\text{-Mod} \to Ab$. By Eilenberg-Watts,

$$H \cong \mathrm{Hom}_{\mathbb{Z}[G]}(M, -)$$

for some $\mathbb{Z}[G]$-module M.

In fact, $M = \mathbb{Z}$ with trivial G-action

$$\left(\sum_g n_g \cdot g\right) \cdot m = \left(\sum_g n_g\right) \cdot m$$

Then

$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong HA$$

$\theta : \mathbb{Z} \to A$ trivial action Determined $\theta(1) = a$

But $\theta(g \cdot 1) = g \cdot \theta(1)$, $\theta(1) = g \cdot \theta(1)$ $\forall g \in G$

So

$$\theta(1) \in \{a \in A ; ga = a \, \forall g \in G\} = HA$$

Conversely if $a \in HA$ then $\theta : \mathbb{Z} \to A$ given by $\theta(n) = na$ is in $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. Conclusion:

$$H^i(G, A) := R^i H(A)$$
$$\cong R^i \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)(A)$$
$$= \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

Ex Let $G = \langle x \mid x^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$

$A = \mathbb{Z} \oplus \mathbb{Z}$ $\quad x \cdot (a,b) = (b,a)$

So $R = \mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^2-1)$

Then $H^i(G, A) = \mathrm{Ext}^i_R(\mathbb{Z}, A)$

$$R \xrightarrow{\varphi_1} R \xrightarrow{\varphi_2} R \xrightarrow{\varphi_1} R \xrightarrow{\theta} \mathbb{Z} \longrightarrow 0$$

$a \longmapsto a(x-1)$ (φ_1)

$a \longmapsto a(x+1)$ (φ_2)

$R = \mathbb{Z}[x]/(x^2-1)$

$\ker \theta = (x-1)R$

$1 \overset{\theta}{\longmapsto} 1$

$x \longmapsto \theta(x) = 1$

Truncate & apply $\mathrm{Hom}(-, A)$

$$0 \longrightarrow \mathrm{Hom}_R(R, A) \xrightarrow{- \circ \varphi_1} \mathrm{Hom}_R(R, A) \xrightarrow{- \circ \varphi_2} \cdots$$

$g \mapsto g(1)$, $\mathrm{Hom}_R(R,A) \cong A \cong \mathbb{Z}^2$

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$

$(a,b) \longmapsto (a,b) \longmapsto (a+b, a+b)$

$(a,b) \longmapsto (b-a, a-b)$

$g(a) \longmapsto g \circ \varphi_1(a)$

$g \longmapsto g((x-1)(a)) = (x-1)g(a)$

$g(1) = (x-1)g(1)$

$H^0 = \{(a,a) ; a \in \mathbb{Z}\} = \mathbb{Z}(1,1) = HA$

$H^1 = \ker/\mathrm{Im} = \dfrac{\{(a,b) ; a = -b\}}{\{(b-a, a-b) ; a,b \in \mathbb{Z}\}} = (0)$

Similarly $H^2 = (0)$, $H^3 = (0)$, ...

So $H^i(G,A) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$

Significance: see A4

$H^1(G,A) \cong \{\text{crossed homomorphisms}\} / \{\text{principal crossed homo.}\}$

In this case, all crossed homomorphisms are principal

ie $f : G \to A$

$f(gh) = f(g) + g \cdot f(h)$ are of the form

$f = f_a : G \to A$ $f_a(g) = g \cdot a - a$ principal

Suppose that $G \circlearrowright A$ A ab grp

Consider all grps H st we have

$1 \to A \xrightarrow{i} H \xrightarrow{\pi} G \to 1$ with $A \trianglelefteq H$ $H/A \cong G$

$\Rightarrow G \circlearrowright A$ via $g \in G$ pick $h \in H$ $\pi(h) = g$ $g \cdot a = hah^{-1} \in A$

Suppose this coincides with our original G-action

$H^2(G,A) =$ all such extension $\{1 \to A \to H \to G \to 1\}$

modulo Yoneda equivalence

$$\begin{array}{ccccccccc} & 1 & \to & A & \to & H & \to & G & \to 1 \\ \sim & & & \downarrow id & & \downarrow \theta & & \downarrow id & \\ & 1 & \to & A & \to & H' & \to & G & \to 1 \end{array}$$

We always have at least one extension namely $A \rtimes_\theta G$ So in our example $H^2(G,A) = 0$

$\Rightarrow 1 \to \mathbb{Z}^2 \to H \to \mathbb{Z}/2\mathbb{Z} \to 1$ where $x = 1$ acts via permuting coordinates

Another example

k field of char. 0

let $\bar{k}$ be alg closure Let $G = \mathrm{Gal}(\bar{k}/k)$ $G \circlearrowright \bar{k}^*$

$\sigma \cdot \lambda = \sigma(\lambda)$

$H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^*)$ is the <u>Brauer group</u> of k, $\mathrm{Br}(k)$.

$\mathrm{Br}(k)$ gives the structure of all finite-dual division rings D over k with $Z(D) = k$

Eg $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ $(\mathbb{R}, \mathbb{H})$

$\cong H^2(Gal(\mathbb{C}/\mathbb{R}), \mathbb{C}^*) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$

$1 \longmapsto -$

<u>Other Theories</u>

Hochschild ~~ho~~ (co)homology

A ring

M A-A bimodule

$HH_i(M) = Tor_i^{A \otimes A^{op}}(A, M)$

$HH^i(M) = Ext_i^{A \otimes A^{op}}(A, M)$

- Local cohomology
- Sheaf cohomology