Tor

R ring, M right R-module, N left S-module (default $S=\mathbb{Z}$)

$$F: R\text{-Mod} \to S\text{-Mod}$$
$$N \longmapsto M \otimes_R N$$

- F is right exact
- F is additive:

$$\mathrm{Hom}_R(N_1,N_2) \longrightarrow \mathrm{Hom}_S(M\otimes N_1, M\otimes N_2)$$
$$f \longmapsto F(f)(m\otimes n) = m\otimes f(n)$$

<u>Def</u> $\mathrm{Tor}_i^R(M,N) := L_iF(N)$

Important remark: Tor measures how close M is to being flat.

Theorem: The following are equivalent:

1) M is flat
2) $\mathrm{Tor}_i^R(M,N)=0 \quad \forall i\geq 1 \quad \forall N \in \mathrm{Ob}(R\text{-Mod})$
3) $\mathrm{Tor}_1^R(M,N)=0 \quad \forall N\in \mathrm{Ob}(R\text{-Mod})$

Proof:

$1\Rightarrow 2$: Take a left R-mod N & a projective resolution $\cdots P_1 \to P_0 \to N \to 0$ exact
$\because M$ is flat $\Rightarrow \to M\otimes P_1 \to M\otimes P_0 \to M\otimes N \to 0$ exact
So $H_i(F(P_\cdot))=0 \;\forall i\geq 1 \Rightarrow \mathrm{Tor}_i^R(M,N)=0 \;\forall i\geq 1$

$2\Rightarrow 3$ ✓

$3\Rightarrow 1$: let $0\to A\to B$ be exact. Now let C be the cokernel of $A\to B$. So $0\to A\to B\to C\to 0$. Apply $M\otimes -$

$$0=\mathrm{Tor}_1^R(M,C) \longrightarrow M\otimes A \to M\otimes B \longrightarrow M\otimes C \to 0$$ exact $\therefore M$ flat ■

In algebraic geometry, Tor is used to give a measure of "intersection" (see Serre's formula).

eg $R=\mathbb{C}[x]$, $M=\mathbb{C}[x]/(f(x))$, $N=\mathbb{C}[x]/(g(x))$
So N has a free resolution:

$$\begin{array}{ccccccccc} & & & & 1 & \longmapsto & 1+(g(x)) & & \\ 0 & \longrightarrow & (g(x)) & \longrightarrow & \mathbb{C}[x] & \to & N & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & \mathbb{C}[x] & & & & & & \\ & & 1 & \longmapsto & g(x) & & & & \end{array}$$

$$0 \longrightarrow \mathbb{C}[x] \xrightarrow{m} \mathbb{C}[x] \xrightarrow{\pi} N \longrightarrow 0 \quad \text{exact}$$

$$m(p(x)) = p(x)g(x)$$

Tensor with M:

$$\longrightarrow 0 \longrightarrow M \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \longrightarrow M \otimes_{\mathbb{C}[x]} \mathbb{C}[x]$$

$$\downarrow$$

chain complex $\qquad M \otimes_{\mathbb{C}[x]} N$

$$\downarrow$$

$$0$$

So $\mathrm{Tor}_i^R(M,N) = H_i(C_\cdot)$

$$C_\cdot: \longrightarrow 0 \longrightarrow 0 \longrightarrow M \otimes \mathbb{C}[x] \longrightarrow M \otimes \mathbb{C}[x] \longrightarrow 0$$

Notice $H_i(C_\cdot) = 0 \ \forall i \geq 2$ So $\mathrm{Tor}_i^R(M,N) = 0 \ \forall i \geq 2$

0^{th} Tor

$$a \otimes p(x) \longmapsto a \otimes p(x)g(x)$$

$$\begin{array}{ccc} M \otimes_{\mathbb{C}[x]} \mathbb{C}[x] & \longrightarrow & M \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \longrightarrow 0 \\ \wr\wr & & *\wr\wr \\ M & \longrightarrow & M \longrightarrow 0 \\ a & \longmapsto & g(x)a \end{array}$$

So $\mathrm{Tor}_0^R(M,N) = \ker/\mathrm{Im} = M/g(x)M = \mathbb{C}[x]/(f(x),g(x))$

$m = \{a(x) + (f(x))\} \qquad g(x)M = \{g(x)a(x) + (f(x))\} = (g(x), f(x))$

If $h(x) = \gcd(f(x), g(x)) \implies \mathrm{Tor}_0(M,N) = \mathbb{C}[x]/(h(x))$

$$\overset{\shortparallel}{M} \otimes_R N$$

Tor_1: $\quad 0 \longrightarrow M \otimes^{*}_{\mathbb{C}[x]} \mathbb{C}[x] \qquad \mathrm{Im} = 0$

So $\mathrm{Tor}_1^R(M,N) = \ker(M \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \xrightarrow{m} M \otimes_{\mathbb{C}[x]} \mathbb{C}[x])$

$$\begin{array}{ccc} \wr\wr & & \wr\wr \\ M & \longrightarrow & M \\ & a \longmapsto ag & \end{array}$$

So $\mathrm{Tor}_1 = \ker(M \to M \ \ a \mapsto ag)$

$M = \mathbb{C}[x]/(f(x)) = \{a(x) + (f(x)) : f(x) \mid a(x)g(x)\}$

$\mathrm{Tor}_1^R(M,N) = \{a(x) + (f(x)) \mid f(x) \mid a(x)g(x)\}$

$= \{a(x) + (f(x)) \mid f(x) \mid a(x)h(x)\}$

Write $f(x) = s(x)h(x) \implies s(x) \mid a(x)$

$$\mathrm{Tor}_1^R(M,N) = (s(x))/(f(x))$$

A4: $\mathrm{Tor}_1^R(R/I, R/J) \cong I \cap J / IJ$

Flatness Criterion

Theorem: Let R be a ring. Let M be a right R-module. TFAE
1) M is flat
2) $M \otimes I \to M = M \otimes_R R$ $\forall$ left ideal $I \subset R$
3) $\mathrm{Tor}_1(M, R/I) = 0$ $\forall I \subset R$ left ideal

Proof:
1 => 2
2 => 3 $\underset{\text{exact}}{0 \to I \xrightarrow{\text{proj.}} R} \to R/J \to 0$

$\Rightarrow 0 \to \mathrm{Tor}_1^R(M, R/I) \to M \otimes I \xrightarrow{1\text{-}1} M \otimes R \to M \otimes R/I \to 0$

$\Rightarrow \mathrm{Tor}_1^R(M, R/I) = 0$

3 => 1 Suppose 3 holds but M is not flat. Then $\exists N' \subseteq N$ left R-module st $M \otimes_R N' \underset{\text{not } 1\text{-}1}{\xrightarrow{id \otimes i = \varphi}} M \otimes_R N$

2016 03 24

Reduction 1: WLOG N' is f.g.
Verification: $\exists x \neq 0$ in $M \otimes N'$ st $\varphi(x) = 0 \in M \otimes N$
Write $x = m_1 \otimes n_1 + \cdots + m_k \otimes n_k$
Let $N_0 \subseteq N'$ be $Rm_1 + \cdots + Rm_k$
Then $x \in M \otimes N_0$ & $\varphi_{M \otimes N_0}$ sends x to 0
Reduction 2: WLOG N is f.g.
Why? Notice $M \otimes N$ is a free-$\mathbb{Z}$-module on symbols (m,n) modulo some relations. So if $x=0$ that means we can capture the fact that $x=0$ using finitely many relations from them. So we take N to be generated by the finitely many relevant things here.
So now we have $N_0 \subseteq N$ both f.g. & $M \otimes N_0 \xrightarrow{\text{not } 1\text{-}1} M \otimes N$
Write $N_0 = \langle n_1, \ldots, n_k \rangle$, $N = \langle n_1, \ldots, n_k, u_1, \ldots, u_m \rangle$
Let $N_i = \langle n_1, \ldots, n_k, u_1, \ldots, u_i \rangle$. So $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = N$
Reduction 3: WLOG $N' = N_i$, $N = N_{i+1}$
Why? $M \otimes N_0 \to M \otimes N$ is $M \otimes N_0 \to M \otimes N_1 \to \cdots \to M \otimes N_m$
so one of them is not 1-1.
Remark: $N_{i+1}/N_i = \dfrac{\langle n_1, \ldots, n_k, u_1, \ldots, u_{i+1} \rangle}{\langle n_1, \ldots, n_k, u_1, \ldots, u_i \rangle}$

so $\exists\, \psi: R \to N_{i+1}/N_i \quad r \mapsto r u_{i+1} + N_i$. Let $I = \ker \psi$

Then $N_{i+1}/N_i \cong R/I$, ie $0 \to N_i \to N_{i+1} \to R/I \to 0$ is exact

$$\Rightarrow \underbrace{\to M\otimes N_i \to M \otimes N_{i+1} \to M \otimes R/I \to 0}_{\substack{\mathrm{Tor}_1^R(M, R/I) \\ \| \\ 0}} \quad \text{is exact}$$

By 3, $\mathrm{Tor}_1^R(M, R/I) = 0 \Rightarrow 0 \to M\otimes N_i \to M \otimes N_{i+1}$ is exact. ∎

Corollary: Let k be a field & let $R = k[t]/(t^2)$ & let M be an R-module. Then M is flat if and only if $M/tM \cong tM$.

Proof: M is flat $\iff \mathrm{Tor}_1^R(M, R/I) = 0 \quad \forall I \trianglelefteq R$

Notice $I = (0)$ or $I = (t)$

$I = 0 \to R/I = R$ projective $\quad \mathrm{Tor}_1^R(M, R) = 0$

So M is flat $\iff \mathrm{Tor}_1^R(M, R/(t)) = 0$

Notice $R/(t) \cong k$, ie $\sim \iff \mathrm{Tor}_1^R(M, k) = 0$

Check

$$0 \to R \to R \xrightarrow{\pi} k \to 0$$
$$r \mapsto rt$$

Now tensor with M

$$\mathrm{Tor}_1^R(M,k) \to M\otimes_R R \to M \otimes_R R \to M \otimes_R k \to 0$$

$\uparrow$ projective

$0 = \mathrm{Tor}_1^R(M, R)$

we just don't know

So

$$0 \to \mathrm{Tor}_1^R(M,k) \to \overset{M}{\overset{\|}{M\otimes_R R}} \to \overset{M}{\overset{\|}{M\otimes R}} \to M\otimes k \to 0 \quad \text{is exact}$$

$M\otimes k = M \otimes R/(t) = M/tM$

So M is flat $\iff \mathrm{Tor}_1^R(M,k) = 0 \iff 0 \to M \xrightarrow{m \mapsto tm} M \to M/tM \to 0$ is exact

$0 \to M\otimes_R R \to M \otimes_R R \to 0 \to$

~~M flat~~ $0 \to tR \to R \xrightarrow{\pi} k \to 0$ exact

M flat $\iff 0 \to M \otimes tR \to M \otimes R \to M/tM \to 0$

$\iff 0 \to M\otimes tR \to M \to M/tM \to 0$

did this get figured out?

2016 03 29

Theorem: Let R be a commutative ring, $a \in R$ not a zero divisor. If M is flat then $am = 0 \Rightarrow m = 0$ for $m \in M$.

Proof:

$$0 \longrightarrow R \xrightarrow{x \mapsto xa} R \longrightarrow R/aR \longrightarrow 0$$

Tensor with M (assume M is flat):

$$0 \longrightarrow \underset{\substack{\cong \\ M}}{M \otimes_R R} \longrightarrow \underset{\substack{\cong \\ M}}{M \otimes_R R} \longrightarrow M \otimes R/aR \longrightarrow 0$$

So the map $M \to M$, $m \mapsto am$ is 1-1. So $am = 0 \Rightarrow m = 0$. ■

The converse is true if R is a PID.

Theorem: If R is a PID then M is flat if and only if M is torsion free, ie if $a \in R \setminus \{0\}$ & $am = 0$ then $m = 0$.

Proof: Suppose M torsion free and let $a \in R \setminus \{0\}$. Then

$$0 \longrightarrow M \xrightarrow{m \mapsto am} M \longrightarrow M/aM \longrightarrow 0 \quad \text{is exact}$$

$$0 \to R \xrightarrow{x \mapsto ax} R \to R/aR \to 0 \qquad \text{tensor with } M$$

$$\mathrm{Tor}_1^R(M,R) \to \mathrm{Tor}_1^R(M,R/aR) \to M \otimes R \xrightarrow[\text{1-1}]{m \mapsto am} M \otimes R \to M \otimes R/aR \to 0$$

But M is torsion free so
$$rm = m \otimes r \longmapsto m \otimes ra = ram$$
$$M \otimes R \longrightarrow M \otimes R \quad \text{is 1-1}$$

So $\mathrm{Tor}_1^R(M, R/aR) = 0$ $\forall a \neq 0$. So $\because$ R is a PID,
$\mathrm{Tor}_1^R(M, R/I) = 0$ $\forall I \trianglelefteq R \Rightarrow M$ is flat. ■

So for example, in $\mathbb{Z}$:
Injectives $\Leftrightarrow$ $\oplus$ $\mathbb{Q}$, C_{p^∞}
projectives $\Leftrightarrow$ free
flat $\Leftrightarrow$ torsion free
In particular, f.g. and flat $\Leftrightarrow$ free

General Facts

Let R be a comm ring and let M, N be R-modules.

$$\begin{array}{c} M \otimes_R N \cong N \otimes_R M \\ \parallel \\ \mathrm{Tor}_0^R(M,N) \cong \mathrm{Tor}_0^R(N,M) \end{array}$$

Fact: $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}_i(N,M)$

Fact: R, S comm, A R-mod, C S-mod, B R- & S-mod

If B is flat as an R-mod and as an S-mod then

$$\mathrm{Tor}_n^S(A \otimes_R B, C) \cong \mathrm{Tor}_n^R(A, B \otimes_S C)$$

$n=0$: $\quad (A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C)$

Special case: S is a flat R-algebra, $B = S$

$$\mathrm{Tor}_n^S(A \otimes_R S, C) \cong \mathrm{Tor}_n^R(A, C)$$

Ext

R ring, M & N left R-mod
Create $\mathrm{Ext}^i_R(M,N)$
Method 1: $G := \mathrm{Hom}(M,-) : R\text{-Mod} \to \mathcal{A}b$ additive & left exact
Then $\mathrm{Ext}^i_R(M,N) = R^iG(N)$

Computing it

$$0 \to N \to I^0 \to I^1 \to \cdots \quad \text{inj. res.}$$

$$0 \to \mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \cdots$$

cochain complex

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}(M, I^\cdot))$$

Eg. $\mathrm{Ext}^i_{\mathbb{Z}}(\underset{M}{\mathbb{Z}/3\mathbb{Z}}, \underset{N}{\mathbb{Z}/3\mathbb{Z}})$

$$0 \to \mathbb{Z}/3\mathbb{Z} \to C_3 \xrightarrow{x \mapsto x^3} C_3 \to 0$$

$$0 \to \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, C_3) \xrightarrow{a} \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, C_3) \xrightarrow{b} 0 \to 0 \to 0$$

$$\psi(x) \longmapsto \psi(x)^3$$

$\psi : \mathbb{Z}/3\mathbb{Z} \to C_3 \xrightarrow{x \mapsto x^3} C_3$ $\quad \psi \in \ker \iff \psi(1)^3 = 1 \in C_3$

$1 \mapsto e^{2\pi i/3}$ $\quad 1 \mapsto e^{2\pi i 2/3}$ $\quad 1 \mapsto 1$

So $\ker(a) \cong \mathbb{Z}/3\mathbb{Z}$ $\quad \mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

$= \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$

$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0 \quad \forall i \geq 2$
We just need to find

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = \ker(b)/\mathrm{Im}(a)$$

$$= \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, C_3)/0 = \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

Alternative description

$$\tilde{G} = \mathrm{Hom}(-, N) : (R\text{-Mod})^{op} \to \mathcal{A}b$$

reverse arrows $\qquad$ left exact & additive

We can compute $R^i\tilde{G}$. To do this, ~~we~~ we use an inj res of M in $(R\text{-Mod})^{op}$

$$0 \to M \to I^0 \to I^1 \to \cdots \quad \text{in } (R\text{-Mod})^{op}$$

$$\cdots \to I^1 \to I^0 \to M \to 0 \quad \text{in } R\text{-Mod}$$

I^i are proj.

So if we take a proj res $\cdots \to P^0 \to M \to 0$ in R-Mod

Apply $\mathrm{Hom}(-, N)$

$$0 \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^1, N) \to \cdots$$

& $R^i\tilde{G}(M) = H^i(\mathrm{Hom}(P^\cdot, N))$

Fact: $R^i\tilde{G}(M) = \mathrm{Ext}^i_R(M, N)$

Ext via Yoneda equivalence

If X, X' are two R-modules &

$$\begin{tikzcd} \alpha: 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "id"] & 0 \\ \alpha': 0 \arrow[r] & A \arrow[r] & X' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

We'll say $\alpha \sim_Y \alpha'$ if $\exists f: X \to X'$ st the diagram commutes

$E^1(A, B)$ = set of equivalence classes of

$$\alpha: 0 \to A \to X \to B \to 0$$

Fact: $E^1(A, B) \cong \mathrm{Ext}^1_R(A, B)$

More generally,

$$\begin{tikzcd} \alpha: 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & X_1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, "f_n"] & B \arrow[r] \arrow[d, "id"] & 0 \\ \alpha': 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & X_1' \arrow[r] \arrow[d, "g_1"] & \cdots \arrow[r] & X_n' \arrow[r] \arrow[d, "g_n"] & B \arrow[r] \arrow[d, "id"] & 0 \\ \alpha'': 0 \arrow[r] & A \arrow[r] & X_1'' \arrow[r] & \cdots \arrow[r] & X_n'' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Take eq. rel. generated by analogous relation.

$E^n(A, B)$ = collection of equiv classes of exact sequences

$$0 \to A \to X_1 \to \cdots \to X_n \to B \to 0, \quad 0 \to A \to Y_1 \to \cdots \to Y_m \to B \to 0$$

Remark: $E^n(A,B) \times E^m(B,C) \to E^{n+m}(A,C)$

$$0 \to A \to X_1 \to \cdots \to X_n \to Y_1 \to \cdots \to Y_m \to 0$$

Take $A = B = C$, $\oplus E^n(A,A)$ is a graded ring

2016 03 31

Ex $R = \mathbb{C}[x]$, $M = \mathbb{C}[x]/(f(x))$, $N = \mathbb{C}[x]/(g(x))$, f, g monic, irred

$$\cdots \to 0 \to R \to R \to M \to 0$$
$$h(x) \overset{\psi}{\longmapsto} h(x)f(x)$$

To compute $\mathrm{Ext}^i_R(M,N)$

$$\theta \longmapsto \theta \circ \psi$$
$$0 \to \mathrm{Hom}_R(R,N) \to \mathrm{Hom}_R(R,N) \to 0 \to \cdots$$

ie $\quad 0 \to N \xrightarrow{\psi} N \to 0 \cdots$

$$n \longmapsto nf(x)$$

$\mathrm{Ext}^0_R(M,N) = H^0(0 \to N \xrightarrow{\psi} N \to 0 \to \cdots) = \ker(\psi) = \begin{cases} 0 & \text{if } f,g \text{ coprime} \\ N = M & \text{if } f = g \end{cases}$

$\mathrm{Ext}^1_R(M,N) = N/\mathrm{Im}\psi = \begin{cases} 0 & \text{if } f,g \text{ coprime} \\ N = M & \text{if } f = g \end{cases}$

$\mathrm{Ext}^i_R(M,N) = 0 \quad \forall i \geq 2$

$$N \xrightarrow{\psi} N$$
$$a(x) + (g(x)) \longmapsto f(x)a(x) + (g(x))$$

If f,g are coprime then $\overline{f(x)} \in \underset{\text{field}}{\underset{\uparrow}{\mathbb{C}[x]/(g(x))}}$ is a unit So ψ isom

if not, $f = g \Rightarrow \psi = 0$

Remark: If R is a PID & M is a f.g. R-module then $\mathrm{Ext}^i_R(M,N) = 0$ $\forall i \geq 2$ $\forall N$

Exercise

Theorem: Let R be a ring & let M be a left R-mod. TFAE

1) M is proj
2) $\mathrm{Ext}^i_R(M,N) = 0 \ \forall i \geq 1 \ \forall N$
3) $\mathrm{Ext}^1_R(M,N) = 0 \ \forall N$

Proof:

$1 \Rightarrow 2$: $0 \to 0 \to M \xrightarrow{id} M \xrightarrow{\varepsilon} 0$ is a proj. res. of M

$\Rightarrow 0 \to \mathrm{Hom}(M,N) \to 0 \to 0 \to \quad \Rightarrow \mathrm{Ext}^i_R(M,N) = 0 \ \forall i \geq 1$

$2 \Rightarrow 3$: ✓

$3 \Rightarrow 1$: $\exists P \overset{epi}{\twoheadrightarrow} M$ proj. Let K denote the kernel

$0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$ Applying $\mathrm{Hom}(-,N)$ gives

$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to$

$\to \mathrm{Ext}^1_R(M,N) \to \cdots$

By assumption, $\mathrm{Ext}^i_R(M,N)=0$

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to 0 \text{ exact } \forall N$$

Remark: To show that M is proj. it suffices to ~~show~~ prove $0 \to K \to P \to M \to 0$ splits since $\Rightarrow P \cong M \oplus K$ & we have $P \oplus Q \cong R^I \Rightarrow M \oplus (Q \oplus K) \cong R^I$ $\Rightarrow M$ proj.

Take $N=K$. $\mathrm{Hom}(P,K) \longrightarrow \mathrm{Hom}(K,K) \ni id_K$

$$\psi \longmapsto \psi \circ f$$

So $\exists \psi$ st $\psi \circ f = id_K$ So $P \cong K \oplus M$ via (ψ, g) ∎

Other version

Theorem: Let R be a ring & let N be a left R-mod. TFAE
1) N is inj
2) $\mathrm{Ext}^n_R(M,N)=0$ $\forall n \geq 1$ $\forall M$
3) $\mathrm{Ext}^1_R(M,N)=0$ $\forall M$

Proof:

$1 \Rightarrow 2$. N inj $\Rightarrow 0 \to N \to N \to 0 \to \ldots$ inj res. Apply $\mathrm{Hom}(M,-)$

$$0 \to \mathrm{Hom}(M,N) \to 0 \to \ldots$$ So (2) ~~that~~ follows.

$2 \Rightarrow 3$: ✓

$3 \Rightarrow 1$: Pick a mono $N \hookrightarrow I$

$$0 \to N \xrightarrow{f} I \xrightarrow{g} K \to 0 \qquad K = \mathrm{coker}(N \to I)$$

Apply $\mathrm{Hom}(M,-)$: $0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,I) \to \mathrm{Hom}(M,K) \to \mathrm{Ext}^1(M,N) = 0$

$M=K$ $\mathrm{Hom}(K,I) \longrightarrow \mathrm{Hom}(K,K)$

$$\exists \psi \longmapsto g \circ \psi = id_K \Rightarrow I = N \oplus K \text{ so } N \text{ is inj.}$$ ∎

Ubiquity of Ext and Tor

Many (co)homology theories one encounters in algebra can really be described using Tor/Ext. This really comes down to results of Eilenberg and Watts.

Theorem [Eilenberg-Watts]: Let R and S be rings (with 1) and let $F: R\text{-Mod} \to S\text{-Mod}$ be right-exact, additive, and preserve small coproducts. (Notice that if M is a left S-module and right R-module then $M \otimes -$ is such a functor.) Then there is an S-R-bimodule M such that $F \cong M \otimes_R -$.

$L_i F \cong \mathrm{Tor}_i(M,-)$

Theorem: If $G: R\text{-Mod} \to S\text{-Mod}$ is left exact, additive, and satisfies

$$G\left(\bigoplus_{i\in I} M_i\right) = \prod_{i\in I} G(M_i)$$

and is contravariant, then there exists an R-S-bimodule M such that $G \cong \mathrm{Hom}_R(-,M)$.

$R^i G \cong \mathrm{Ext}^i_R(-,M)$

Theorem: If $H: R\text{-Mod} \to \mathrm{Ab}$ is left-exact, additive, covariant, and commutes with projective limits ($\varprojlim$), then there exists a left R-module M such that $H \cong \mathrm{Hom}_R(M,-)$.

$R^i H \cong \mathrm{Ext}^i(M,-)$