PMATH 945 - Topics in Algebra: Category Theory and Homological Algebra

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Classes vs Sets: Classes = Sets $\dot{\cup}$ Proper Classes

Categories

$\mathcal{C}$ = a category has two parts:

$Ob(\mathcal{C})$ = a class of objects

& for each pair $A, B \in Ob(\mathcal{C})$ $\exists$ a set of morphisms $Hom_{\mathcal{C}}(A,B)$

We'll write $f: A \to B$ to mean $f \in Hom_{\mathcal{C}}(A,B)$

We have a composition law

$$\circ : Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

& composition is associative (when defined) $f \circ (g \circ h) = (f \circ g) \circ h$

Finally, $\forall A \in Ob(\mathcal{C})$ $\exists id_A \in Hom_{\mathcal{C}}(A,A)$ $id_A \circ f = f$, $g \circ id_A = g$ when defined.

Examples

$\mathcal{G}rp$ = Category of groups

Objects = class of all groups

If G, H are groups then $Hom_{\mathcal{G}}(G,H) = \{f: G \to H ; f \text{ is a group homomorphism}\}$

Notice we have composition

& $id_G : G \to G$

Ex $\mathcal{S}et$ = Category of Sets

$Ob(\mathcal{S}et)$ = class of all sets

$Hom_{\mathcal{S}et}(X,Y) = \{f: X \to Y\}$

Ex Top = category of topological spaces

Ob = topological spaces

$Hom_{Top}(X,Y) = \{f: X \to Y ; f \text{ is cts}\}$

Ab = category of abelian groups

base point

Top^* = pointed topological spaces $(X, x_0) \xrightarrow{f} (Y, y_0)$ cts, $f(x_0) = y_0$

Important example for sheaves
X = topological space
Top_X
Objects = $\{U;\ U \subseteq X \text{ is open}\}$
$$\mathrm{Hom}_{Top_X}(U,V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V \\ \{i: U \to V\} & \text{if } U \subseteq V \end{cases}$$

Why?
Categories can provide a unification tool.

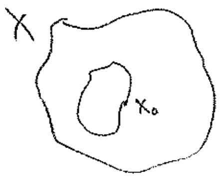
Functors
Let $\mathcal{C}$, $\mathcal{D}$ be two categories
$$F: \mathcal{C} \to \mathcal{D}$$
is called a functor from $\mathcal{C}$ to $\mathcal{D}$ with the following properties
1) $\forall A \in Ob(\mathcal{C})\ \exists F(A) \in Ob(\mathcal{D})$, ie $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$
2) $\forall f: A \to B$, $A, B \in Ob(\mathcal{C})$ $\exists F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, $F(f): F(A) \to F(B)$
respecting composition & identity:
$$F(id_A) = id_{F(A)}$$
$$f \circ g = h \Rightarrow F(f) \circ F(g) = F(h)$$

Ex $F: \mathcal{A}b \to \mathcal{G}rp$
$F(A) = A$
$F(f) = f$

Ex $T: \mathcal{G}rp \to \mathcal{A}b$
$T(G) = G/G'$
$f: G \to H$
$T(f): G/G' \to H/H'$
$T(f)(gG') = f(g)H'$

Top^* = pointed topological spaces (X, x_0)
Functor: $\pi_1 : Top^* \to \mathcal{G}rp$
$(X, x_0) \mapsto \pi_1(X, x_0)$

What is it?

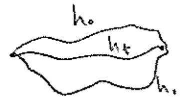


loops: $h : [0,1] \to X$ cts, $h(0) = h(1) = x_0$

$\pi_1(X, x_0) = $ loops based at $x_0 / \sim$

$\sim$ = homotopy equivalence

$h_0 \sim h_1 \iff \exists h_t \ \forall t \in [0,1]$ $\quad h_0 = h_0,\ h_1 = h_1$



$h : [0,1]^2 \to X$ cts

$(x, t) \longmapsto h_t(x)$

$f : (X, x_0) \longrightarrow (Y, y_0)$

$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

$\pi_1(f)(g) = f \circ g : [0,1] \to Y$

<u>Ex</u> $F : \mathcal{G}rp \longrightarrow Set$ "forgetful functor"

<u>Natural Transformations</u>

$\mathcal{C}, \mathcal{D}$ categories

$F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors

A map $\alpha : F \to G$ is called a <u>natural transformation</u> if:

1) $\forall A \in Ob(\mathcal{C}) \ \exists \alpha_A : F(A) \to G(A)$, ie $\alpha_A \in Hom_{\mathcal{D}}(F(A), G(A))$ & whenever $f : A \to B$, $A, B \in Ob(\mathcal{C})$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$

$$\begin{tikzcd} & F(A) \in Ob(\mathcal{D}) \arrow[d, "\alpha_A \in Hom_{\mathcal{D}}(F(A), G(A))"] \\ Ob(\mathcal{C}) \ni A \arrow[ur, "F"] \arrow[r, "G"'] & G(A) \in Ob(\mathcal{D}) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

<u>Remark</u>

If $\exists$ natural transformations $\alpha : F \to G$, $\beta : G \to F$ st $\alpha \circ \beta : G \to G$ is the identity & $\beta \circ \alpha : F \to F$ is the identity, Then we say the functors F and G are isomorphic

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<u>Ex</u> $F : \mathcal{C} \to \mathcal{D}$, $\alpha = id : F \to F$, $\alpha_A : F(A) \to F(A)$, $\alpha_A = id_A$

<u>Ex</u> <u>Double duals</u>

Let $\mathcal{C}$ be the category of finite-dimensional $\mathbb{C}$-v.s. Let $F : \mathcal{C} \to \mathcal{C}$ be the identity functor: $F(V) = V$, $T : V \to W$, $F(T) = T : V = F(V) \to W = F(W)$.

$G: \mathcal{C} \to \mathcal{C}$, $G(V) = V^{**}$, $\alpha: F \to G$ $\forall V \in Ob(\mathcal{C})$ $\alpha_V: F(V) = V \to G(V) = V^{**}$

If $v \in V$ Define $\alpha_V(\vec{v}) = e_{\vec{v}}$, $e_{\vec{v}} \in V^{**}$, $e_{\vec{v}} \in Hom_{\mathbb{C}}(V^*, \mathbb{C})$, $e_{\vec{v}}: V^* \to \mathbb{C}$

$f \in V^*$, $f: V \to \mathbb{C}$, $e_{\vec{v}}(f) = f(\vec{v})$

$T: V \to W$

$$\begin{tikzcd} V \arrow[d, phantom, "\shortparallel"] & V^{**} \arrow[d, phantom, "\shortparallel"] \\ F(V) \arrow[r, "\alpha_V"] \arrow[d, "T = F(T)"'] & G(V) \arrow[d, "G(T)"] \\ F(W) \arrow[r, "\alpha_W"] \arrow[d, phantom, "\shortparallel"] & G(W) \arrow[d, phantom, "\shortparallel"] \\ W & W^{**} \end{tikzcd}$$

What is $G(T)$? $T: V \to W$, $G(T): V^{**} \to W^{**}$, $G(T)(e_{\vec{v}}) := e_{T(\vec{v})}$ (or $G(T) = T^{**}$)

So $\alpha: F \to G$ is a natural transformation

Opposite Category & Adjoints

$\mathcal{C}$ = category.

We have an <u>opposite category</u> $\mathcal{C}^{op}$

$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

$\forall A, B \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$

$f: A \to B$ in $\mathcal{C}$ $\rightsquigarrow$ $\tilde{f}: B \to A$ in $\mathcal{C}^{op}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "f \circ g"'] & B \arrow[r, "g"] & C \end{tikzcd} \quad \text{in } \mathcal{C}$$

$$\begin{tikzcd} A & B \arrow[l, "\tilde{f}"'] & C \arrow[l, "\tilde{g}"'] \arrow[ll, bend left, "\tilde{f} \circ \tilde{g}"] \end{tikzcd} \quad \text{in } \mathcal{C}^{op}$$

& identies are the same

<u>Ex</u>

If $\mathcal{C}$ = category of finite-dimensional $\mathbb{C}$-vector spaces

Then $F: \mathcal{C} \to \mathcal{C}^{op}$ $F(V) = V^*$ $T: V \to W$ $F(T) = T^*: W^* \to V^*$

$G: \mathcal{C}^{op} \to \mathcal{C}$ $G(V) = V^*$ $T: V \to W$ $G(T) = T^*: W^* \to V^*$

$G \circ F: \mathcal{C} \to \mathcal{C}$ $V \to V^{**}$ $T: V \to W$ $T^{**}: V^{**} \to W^{**}$

$F \circ G: \mathcal{C}^{op} \to \mathcal{C}^{op}$ sends $V \to V^{**}$.

Exercise: $G \circ F$ is naturally isomorphic to the identity functor from $\mathcal{C} \to \mathcal{C}$

<u>Def</u> If $\mathcal{C}$ and $\mathcal{D}$ are categories & $F: \mathcal{C} \to \mathcal{D}$ $G: \mathcal{D} \to \mathcal{C}$ such that $F \circ G: \mathcal{D} \to \mathcal{D}$ and $G \circ F: \mathcal{C} \to \mathcal{C}$ are isomorphic to the respective identity functors. Then we say $\mathcal{C} \cong \mathcal{D}$ are equivalent.

Ex (algebraic geometry)

$\mathcal{C}$ = category of finitely generated, reduced $\mathbb{C}$-algebras

Let k be a field. A k-algebra B is just a commutative ring with an injective homomorphism $\varphi: k \to B$, $1_k \to 1_B$

Ex $B = \mathbb{C}[x,y]$ is a $\mathbb{C}$-algebra, $\varphi: \mathbb{C} \to B$ $\varphi(\lambda) = \lambda$

B is finitely generated as a k-algebra if $\exists a_1, \dots, a_d \in B$ such that every $b \in B$ can be written as a polynomial $p(a_1, \dots, a_d)$, $p(X_1, \dots, X_d) \in k[X_1, \dots, X_d]$.

B is reduced means if $b \in B$ & $b^n = 0$ for some $n \geq 1 \Rightarrow b = 0$

eg $\mathbb{C}[x]/(x^2)$ is not reduced, $\mathbb{C}[X_1, X_2, X_3, \dots]$ is not finitely generated

$\mathcal{D}$ = complex affine varieties

objects are $Y \subseteq \mathbb{C}^n$ for some $n \geq 1$, Y = zero set of a finite set of polynomials $p_1(X_1, \dots X_n), \dots, p_d(X_1, \dots X_n)$

Ex $\{(a,b) \in \mathbb{C}^2 ; b^2 = a^3 + 1\} \subseteq \mathbb{C}^2$, Y = zero set $x_2^2 - x_1^3 - 1$

Algebraic geometry: $\mathcal{C} \cong \mathcal{D}^{op}$

$F: \mathcal{C} \to \mathcal{D}^{op}$

Nullstellensatz

$$\begin{array}{l} B \in \mathcal{C}, \quad B \cong \mathbb{C}[X_1, \dots X_n]/(p_1(X_1, \dots X_n), \dots, p_d(X_1, \dots, X_n)) \\ \quad\quad F \downarrow \\ \quad\quad\quad Y = \text{zero set in } \mathbb{C}^n \text{ of } p_1, \dots p_d \end{array}$$

$G: \mathcal{D}^{op} \to \mathcal{C}$

$$\begin{array}{l} \quad Y \to \mathbb{C}[X_1, \dots X_n]/(p_1, \dots p_d) \\ \quad \shortparallel \\ Z(p_1, \dots, p_d) \end{array}$$

$p_1, \dots, p_d \in \mathbb{C}[X_1, \dots X_d]$

Adjoint Functors

Def Let $\mathcal{A}, \mathcal{B}$ be two categories. We say that $F: \mathcal{A} \to \mathcal{B}$ is a left adjoint for $G: \mathcal{B} \to \mathcal{A}$ if, intuitively,

$$\mathrm{Hom}_{\mathcal{A}}(A, G(B)) \cong \mathrm{Hom}_{\mathcal{B}}(F(A), B) \quad \forall A \in Ob(\mathcal{A}), B \in Ob(\mathcal{B})$$

More formally $\forall A \in Ob(\mathcal{A})$, $B \in Ob(\mathcal{B})$ $\exists \alpha_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, G(B)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(F(A), B)$

So F is the left adjoint, G is the right adjoint, such that whenever $\varphi: \mathrm{Hom}_{\mathcal{A}}(A, A')$ $A \xrightarrow{\varphi} A'$ $A, A' \in Ob(\mathcal{A})$, $\psi: B \to B'$, $B, B' \in Ob(\mathcal{B})$

Then

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(A', G(B)) \arrow[r, "\alpha_{A',B}"] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(\varphi, G(\psi))"'] & \mathrm{Hom}_{\mathcal{B}}(F(A'), B) \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(F(\varphi), \psi)"] \\
\mathrm{Hom}_{\mathcal{A}}(A, G(B')) \arrow[r, "\alpha_{A,B'}"] & \mathrm{Hom}_{\mathcal{B}}(F(A), B')
\end{tikzcd}$$

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$F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{A}$ we write

$$F \rightleftarrows G$$

to mean F is a left adjoint for G & G is a right adjoint for F.

What are the vertical maps?

$\mathrm{Hom}_{\mathcal{A}}(\varphi, G(\psi)) : \mathrm{Hom}(A', G(B)) \to \mathrm{Hom}(A, G(B'))$

Given $f: A' \to G(B) \rightsquigarrow \hat{f}. A \to G(B')$

$$A \xrightarrow{\varphi} A' \xrightarrow{f} G(B) \xrightarrow{G(\psi)} G(B')$$

So $\mathrm{Hom}_{\mathcal{A}}(\varphi, G(\psi))(f) = G(\psi) \circ f \circ \varphi$

<u>Ex</u> 1) $\mathcal{G}$ = category of groups, $\mathcal{A}$ = category of abelian groups

$I: \mathcal{A} \to \mathcal{G}$, $I(A) = A$, $A \xrightarrow{f} B \rightsquigarrow A \xrightarrow{I(f) = f} B$

$R: \mathcal{G} \to \mathcal{A}$, $R(G) = G/G'$. $f: G \to H \rightsquigarrow G/G' \xrightarrow{R(f)} H/H'$

<u>Ex</u> $\mathcal{A}$: category of abelian groups, $\mathcal{S}et$ = category of sets

We have a "forgetful" functor $G: \mathcal{A} \to \mathcal{S}et$, $G(A) = A$, $G(f) = f$.

Consider $F: \mathcal{S}et \to \mathcal{A}$, $F(X) = \mathbb{Z}^X := \bigoplus_{x \in X} \mathbb{Z}$, $x \in X \rightsquigarrow e_x$

$$\{\textstyle\sum_{x \in X} n_x e_x : n_x = 0 \text{ for all but finitely many } x \in X\}$$

Then $F \rightleftarrows G$; if X is a set, A abelian group

$$\mathrm{Hom}_{set}(X, \underset{A}{\underset{\parallel}{G(A)}}) \underset{\alpha_{X,A}}{\cong} \mathrm{Hom}_{\mathcal{A}}(F(X), A)$$

$f: X \to A$ $\hat{f}: \mathbb{Z}^X \to A$, $e_x \mapsto f(x)$

$\varphi: X \xrightarrow{set} X'$, $\psi: A \to A'$

$$\begin{tikzcd}
\mathrm{Hom}_{set}(X', G(A)) \arrow[r, "\alpha_{X',A}"] \arrow[d, "\mathrm{Hom}_{set}(\varphi, G(\psi))"] & \mathrm{Hom}_{\mathcal{A}}(F(X'), A) \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(F(\varphi), \psi)"] \\
\mathrm{Hom}_{set}(X, G(A')) \arrow[r, "\alpha_{X,A'}"] & \mathrm{Hom}_{\mathcal{A}}(F(X), A')
\end{tikzcd}$$

Really, this is:

$$\begin{tikzcd}
& f \arrow[rr, mapsto] & & \tilde{f} & \\
f \in \arrow[d, mapsto] & \mathrm{Hom}_{set}(X', A) \arrow[r, "\alpha_{X',A}"] \arrow[d] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}^{X'}, A) & \hat{f} \arrow[d, mapsto] \\
\psi \circ f \circ \varphi & \mathrm{Hom}_{set}(X, A') \arrow[rr, "\alpha_{X,A'}"] & & \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}^X, A') & \psi \circ f \circ \varphi \\
& \psi \circ f \circ \varphi \arrow[rr, mapsto] & & \psi \circ \tilde{f} \circ \hat{\varphi} & =
\end{tikzcd}$$

Exercise: Stone-Čech compactification

Idea: CHaus = Category

Objects = compact Hausdorff spaces

$f: X \to Y$, cts maps

Top = category of topological spaces

$G: \mathcal{C}Haus \to Top$, $G(X) = X$, $X \xrightarrow{f} Y$ $G(f) = f$.

In other words, CHaus is a subcategory of Top.

i.e. $Ob(\mathcal{C}Haus) \subseteq Ob(Top)$

$Hom_{CHaus}(X,Y) \subseteq Hom_{Top}(X,Y) \quad \forall X, Y \in Ob(\mathcal{C}Haus)$

& $\underset{CHaus}{f \circ g} = \underset{Top}{f \circ g}$ when it makes sense

$(id_X)_{CHaus} = (id_X)_{Top}$ for X compact Hausdorff space

What would a left adjoint do?

$F: Top \to \mathcal{C}Haus$

$X \rightsquigarrow F(X)$ compact Hausdorff

$$Hom_{Top}(X, \underset{G(Y)}{F(X)}) \overset{\alpha_{X,F(X)}}{\cong} Hom_{CHaus}(F(X), \underset{Y}{F(X)})$$

Take $\beta = \alpha^{-1}_{X,F(X)}(id_{F(X)})$, $\beta: X \to F(X)$

Moreover the adjoint property shows that if $f: X \to K$ cts, K cpt Hausdorff

$\exists! \tilde{f}: F(X) \to K$

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "f"'] & F(X) \arrow[dl, "\tilde{f}"] \\ K & \end{tikzcd}$$

Example: $Top^* =$ pointed topological spaces (X, x_0)

$\mathcal{G}rp$ = category of groups

$\pi_1: Top^* \to \mathcal{G}rp \quad (X, x_0) \mapsto \pi_1(X, x_0)$

Two examples

$(X, x_0) = (\mathbb{C}, 1) \quad \pi_1(X, x_0) = \{id\}$

$g: [0,1] \xrightarrow{cts} \mathbb{C}$ $g(0) = g(1) = 1$, $g_t(x) = g(x)t + 1(1-t)$, $g_1 = g$, $g_0 = 1$

$(Y, y_0) = (S^1, 1)$

Suppose that π_1 has a left adjoint $F: \mathcal{G}rp \to Top^*$

$H = \mathbb{Z} \in Ob(\mathcal{G}rp)$

$$\mathrm{Hom}_{Grp}(H, \overset{\text{size } 1}{\pi_1(X,x_0)}) \cong \mathrm{Hom}_{Top^*}(F(H), \overset{\text{size } 1}{(X,x_0)})$$

$$H = \mathbb{Z}, \quad \pi_1(X,x_0) = \mathbb{Z}$$

$$\mathrm{Hom}_{Grp}(H, \pi_1(Y,y_0)) \cong \underset{\text{infinite}}{\mathrm{Hom}_{Top^*}(F(H), (Y,y_0))}$$

$$H \cong \mathbb{Z}$$

So π_1 does not have a left adjoint.

General principle:
Forgetful functors $\mathcal{A} \to \mathcal{S}et$ are right adjoints to "free" functors $F: \mathcal{S}et \to \mathcal{A}$

What is a "free" object in $\mathcal{A}$?
Given a category $\mathcal{A}$ & a set X, we say $F(X)$ is the <u>free object</u> in X in $\mathbb{A}$ If $\exists$ a set map $f: X \to F(X)$ st if $g: X \to A$ is a set map to some $A \in Ob(\mathbb{A})$ Then $\exists! \tilde{g}. F(X) \to A$, $\tilde{g} \in \mathrm{Hom}_{\mathcal{A}}(F(X), A)$ st

$$\begin{tikzcd} F(X) \arrow[r, "\tilde{g}"] & A \\ X \arrow[u, "f"] \arrow[ur, "g"'] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] \end{tikzcd}$$

Check: If free objects exist $F \rightleftarrows G$ = forgetful functor

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$F \rightleftarrows G$ $\qquad \varphi: A \to A'$ in $\mathcal{A}$
$F: \mathcal{A} \to \mathcal{B}$ $\qquad \psi: B \to B'$ in $\mathcal{B}$
$G: \mathcal{B} \to \mathcal{A}$

(★) ——

$$\begin{tikzcd} f \in & \mathrm{Hom}(A', GB) \arrow[r, "\alpha_{A',B}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(FA', B) \arrow[d] & \ni g \\ G(\psi) \circ f \circ \varphi & \mathrm{Hom}(A, GB') \arrow[r, "\alpha_{A,B'}"] & \mathrm{Hom}(FA, B') & \psi \circ g \circ F(\varphi) \end{tikzcd}$$

Tensor-Hom adjunction... later

Theorem: Right adjoint are unique up to natural isomorphism (same proof works for left adjoints).

ie. If $F: \mathcal{A} \to \mathcal{B}$ & $G, G': \mathcal{B} \to \mathcal{A}$ are two right adjoints for F.
Then $\exists \eta: G \to G'$, $\mu: G' \to G$ natural transformations such that

$\mu \circ \eta = id_G : G \to G$, $\eta \circ \mu = id_{G'} : G' \to G'$
$\forall D \in Ob(\mathcal{B})$ $\exists \eta_D : GD \to G'D$
If $f : D \to D'$, $D, D' \in Ob(\mathcal{B})$

$$\begin{tikzcd}
GD \arrow[r, "\eta_D"] \arrow[d, "Gf"'] \arrow[dr, phantom, "\circlearrowright"] & G'D \arrow[d, "G'f"] \\
GD' \arrow[r, "\eta_{D'}"] & G'D'
\end{tikzcd}$$

Remark: If G' is a right adjoint for F & ~~and~~ G is naturally isomorphic to G', then G is also a right adjoint

<u>Why?</u>
Suppose that $\eta : G \to G'$ is a natural isomorphism.
By assumption

$$f \longmapsto \eta_B \circ f$$

$$\begin{tikzcd}
f \in \arrow[d, mapsto] & Hom(A', GB) \arrow[r] \arrow[d] & Hom(A', G'B) \arrow[r, "\alpha_{A',B}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Hom(FA', B) \arrow[d] \\
G(\psi) \circ f \circ \varphi & Hom(A, GB') \arrow[r] & Hom(A, G'B') \arrow[r, "\alpha_{A,B'}"] & Hom(FA, B')
\end{tikzcd}$$

$$h \longmapsto \eta_{B'} \circ h$$

$\varphi : A \to A'$ $\qquad \llcorner\!\!\rightarrow : f \mapsto G(\psi) \circ f \circ \varphi \mapsto \eta_{B'} \circ G(\psi) \circ f \circ \varphi$
$\psi : B \to B'$ $\qquad \urcorner\!\!\downarrow : f \mapsto \eta_B \circ f \mapsto G(\psi) \circ \eta_B \circ f \circ \varphi$

Let $F : \mathcal{A} \to \mathcal{B}$ & suppose that $G, G' : \mathcal{B} \to \mathcal{A}$ both right adjoints for F.
Fix $A \in Ob(\mathcal{A})$, $D \in Ob(\mathcal{B})$

$$Hom_{\mathcal{A}}(A, GD) \xrightarrow{\alpha_{A,D}} Hom_{\mathcal{B}}(FA, D) \xleftarrow{\alpha'_{A,D}} Hom_{\mathcal{A}}(A, G'D)$$

Take $A = GD$

$$\begin{array}{ccccc}
Hom_{\mathcal{A}}(GD, GD) & \xrightarrow{\alpha_{GD,D}} & Hom_{\mathcal{B}}(FGD, D) & \xleftarrow{\alpha'_{GD,D}} & Hom_{\mathcal{A}}(GD, G'D) \\
id_{GD} & \longmapsto & \alpha_{GD,D}(id_{GD}) & \longmapsto & (\alpha'_{GD,D})^{-1}(\alpha_{GD,D}(id_{GD}))
\end{array}$$

Define $\eta_D : GD \to G'D$ To be $(\alpha'_{GD,D})^{-1}(\alpha_{GD,D}(id_{GD}))$

Must show $f: D \to D'$

$$\begin{tikzcd}
GD \arrow[r, "\eta_D"] \arrow[d, "Gf"'] \arrow[dr, phantom, "\circlearrowright"] & G'D \arrow[d, "G'f"] \\
GD' \arrow[r, "\eta_{D'}"] & G'D'
\end{tikzcd}$$

We'll apply the commuting square (*) twice.
First time $A = A' = GD$, $B = D$, $B' = D'$
$\varphi: A \to A'$, $\varphi = id_{GD}$, $\psi: D \to D'$, $f: D \to D'$

$$\begin{tikzcd}
\mathrm{Hom}(\overset{A'}{GD}, \overset{GB}{GD}) \arrow[r, "\alpha_{GD,D}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\overset{FA'}{FGD}, \overset{B}{D}) \arrow[d] & \mathrm{Hom}(\overset{A'}{GD}, \overset{G'B}{G'D}) \arrow[l, "\alpha'_{GD,D}"'] \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
\mathrm{Hom}(\underset{A}{GD}, \underset{GB'}{GD'}) \arrow[r, "\alpha_{GD,D'}"] & \mathrm{Hom}(\underset{FA}{FGD}, \underset{B'}{D'}) & \mathrm{Hom}(\underset{A}{GD}, \underset{G'B'}{G'D'}) \arrow[l, "\alpha'_{GD,D'}"']
\end{tikzcd}$$

$\rightarrow\downarrow$: $id_{GD} \mapsto \eta_D \mapsto G'(f) \circ \eta_D$
$\downarrow\rightarrow$: $id_{GD} \mapsto G(f) \mapsto \psi \circ G(f)$. $\quad \psi = (\alpha'_{GD,D'})^{-1} \circ \alpha_{GD,D'}$

Apply (*) again
$A = GD$, $A' = GD'$, $\varphi = Gf: GD \to GD'$, $B = B' = D'$, $\psi = id_{D'}$

$$\begin{tikzcd}
\mathrm{Hom}(\overset{A'}{GD'}, \overset{GB}{GD'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\overset{FA'}{FGD'}, D') \arrow[d] & \mathrm{Hom}(\overset{A'}{GD'}, \overset{G'B}{G'D'}) \arrow[l] \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
\mathrm{Hom}(\underset{A}{GD}, \underset{GB'}{GD'}) \arrow[r] & \mathrm{Hom}(FGD, D') & \mathrm{Hom}(\underset{A}{GD}, \underset{G'B'}{G'D'}) \arrow[l]
\end{tikzcd}$$

Start with $id_{GD'}$:
$\rightarrow\downarrow$: $id_{GD'} \mapsto \eta_{D'} \mapsto \eta_{D'} \circ G(f)$
$\downarrow\rightarrow$: $id_{GD'} \mapsto id_{GD'} \circ G(f) \mapsto \psi \circ G(f)$

Square 1 gives: $\psi \circ G(f) = G'(f) \circ \eta_D$
Square 2 gives: $\psi \circ G(f) = \eta_{D'} \circ G(f)$

$$\begin{tikzcd}
GD \arrow[r, "\eta_D"] \arrow[d, "Gf"'] & G'D \arrow[d, "Gf"] \\
GD' \arrow[r, "\eta_{D'}"] & G'D'
\end{tikzcd}$$

Tensor-Hom adjunction

Let A be a commutative ring

A-Mod = category of A-modules

$\mathrm{Hom}_A(M,N) = \mathrm{Hom}_{A\text{-Mod}}(M,N) = \{f: M \to N;\ f \text{ is an } A\text{-module homomorphism}\}$

Fix an A-module M. $F: A\text{-Mod} \to A\text{-Mod}$, $N \overset{F}{\longmapsto} M \otimes_A N$

Universal Property: If P is an A-module & $f: M \times N \xrightarrow{f} P$ is bilinear

$\exists!\ \tilde{f}: M \otimes_A N \to P$ A-module homomorphism

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "i"'] & P \\ M \otimes_A N \arrow[ur, "\exists!\tilde{f}"'] & \end{tikzcd}$$

$i(m,n) = m \otimes n$

$f: N \to N'$ $\quad M \otimes_A N \xrightarrow{id \otimes f = F(f)} M \otimes_A N'$, $m \otimes n \longmapsto m \otimes f(n)$

We also have $G: A\text{-Mod} \to A\text{-Mod}$, $G(N) = \mathrm{Hom}_A(M,N)$

$N \xrightarrow{f} N'$, $\mathrm{Hom}_A(M,N) \xrightarrow{G(f)} \mathrm{Hom}_A(M,N')$

$$\psi \longmapsto f \circ \psi$$

Theorem [Tensor-Hom adjunction]: $F \rightleftarrows G$

$\mathrm{Hom}(FN, B) \cong \mathrm{Hom}(N, GB)$

$\mathrm{Hom}(M \otimes N, B) \cong \mathrm{Hom}(N, \mathrm{Hom}(M,B))$

$\psi: N \to \mathrm{Hom}(M,B)$

$\psi(n): M \to B$

$\psi(n)(m) \in B$

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Now R is the ring, instead of A...

If A & B are two R-modules

$$\mathrm{Hom}_R(A, GB) \xrightarrow{\alpha_{A,B}} \mathrm{Hom}_R(FA, B)$$

$$\| \qquad\qquad\qquad \|$$

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(M,B)) \to \mathrm{Hom}_R(M \otimes_R A, B)$$

Construction: Take $\psi \in \mathrm{Hom}_R(A, \mathrm{Hom}_R(M,B))$. Then for $a \in A$, $\psi(a): M \to B$

$\psi(a)(m) \in B$. Define $\psi_0: M \times A \to B$, $\psi_0(m,a) = \psi(a)(m)$. Fix a.

$\psi_0(rm + m', a) = \psi(a)(rm + m') = r\psi(a)(m) + \psi(a)(m') = r\psi_0(m,a) + \psi_0(m',a)$.

Fix $m \in M$.

$\psi_0(m, ra + a') = [\psi(ra + a')](m) = [r\psi(a) + \psi(a')](m) = r\psi_0(m,a) + \psi_0(m,a')$

By universal property for tensor products

$$\begin{tikzcd} \psi_0 : M \times A \arrow[r] \arrow[d, "{(m,a) \mapsto m\otimes a}"'] & B \\ M \otimes_R A \arrow[ur, "\exists!\, \hat{\psi}_0"'] & \end{tikzcd}$$

So $\alpha_{A,B}(\psi) = \hat{\psi}$.

This is reversible. If $\varphi : M \otimes_R A \to B$. Then $\tilde{\varphi} : M \times A \to B$.

$\tilde{\varphi}(m,a) = \varphi(m \otimes a)$ is bilinear

$$\tilde{\varphi}(rm_1 + m_2, a) = \cdots = r\tilde{\varphi}(m_1,a) + \tilde{\varphi}(m_2,a)$$

Other side is the same

$$\varphi \rightsquigarrow \tilde{\varphi} : M \times A \longrightarrow B \quad R\text{-bilinear}$$

$$\updownarrow$$

$$a \longmapsto \tilde{\varphi}(a) : M \to B \quad R\text{-linear}$$

$$a \longmapsto \hat{\varphi}(a) \in \mathrm{Hom}_R(M,B)$$

$$\tilde{\varphi} \in \mathrm{Hom}(A, \mathrm{Hom}_R(M,B))$$

So $\alpha_{A,B}$ is an isomorphism

$$A \xrightarrow{\varphi} A'$$
$$B \xrightarrow{\psi} B'$$

$$\begin{tikzcd} h \in \mathrm{Hom}(A',GB) \arrow[r, "\alpha_{A',B}"] \arrow[d] & \mathrm{Hom}(FA',B) \arrow[d] & g \arrow[d, mapsto] \\ G(\psi)\circ h \circ \varphi \quad \mathrm{Hom}(A,GB') \arrow[r, "\alpha_{A,B'}"] & \mathrm{Hom}(FA,B') & \psi \circ g \circ F(\varphi) \end{tikzcd}$$

Take $h \in \mathrm{Hom}(A', \mathrm{Hom}(M,B))$

$$h \overset{\alpha_{A',B}}{\longmapsto} \hat{h}_0 \overset{\psi\circ\,\cdot\,\circ F(\varphi)}{\longmapsto} \psi \circ \hat{h}_0 \circ F(\varphi) = \psi \circ \hat{h}_0 \circ (id \otimes \varphi)$$

$$\theta \longmapsto \psi \circ \theta$$

$$\mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(M,B')$$

$$h \overset{G(\psi)\circ\,\cdot\,\circ\varphi}{\longmapsto} G(\psi)\circ h \circ \varphi = \psi \circ h \circ \varphi \overset{\alpha_{A,B'}}{\longmapsto} \widehat{(\psi\circ h\circ\varphi)}_0$$

Must check $\widehat{(\psi\circ h\circ\varphi)}_0 = \psi \circ \hat{h}_0 \circ (id \otimes \varphi)$.

Hint: just look at what they do to $m \otimes a$