

Lawvere Theories and Morita Equivalence

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The eventual goal of this topic is to characterize, for a given variety $\mathcal{V}$, which varieties are equivalent to it. The answer will be to take all Lawvere theories which are Morita equivalent to the Lawvere theory of $\mathcal{V}$, and then take the models of these. A good starting point then is to characterize those Lawvere theories which are Morita equivalent to a given Lawvere theory $\mathbb{T}$. This intermediate step is accomplished in a purely categorical fashion. Unfortunately we won't have time to explore the whole story in detail. For the most part, we follow the path of [1].

Definitions and Background

DEFINITION 1. A [variety](#) $\mathcal{V}$ is a class of algebraic structures with a given signature which satisfy a given set of identities. Equivalently, it is a class of algebraic structures with a given signature which are closed under taking homomorphic images, substructures, and products.

Given an algebraic structure $A \in \mathcal{V}$ and a basic n -ary function symbol σ in the signature of $\mathcal{V}$, we denote by $\sigma^A : A^n \rightarrow A$ the A -interpretation of σ .

EXAMPLE 2. The class of rings is a variety. The signature here consists of two nullary functions (also known as constants) 0 and 1 , one unary operation $-$, and two binary operations $+$ and $\cdot$. The set of identities here consists simply of the ring axioms. In particular, for a ring R , 0^R is the zero element of the ring and $+^R : R^2 \rightarrow R$ is the addition operation of the ring.

There are a host of other important examples, including (but not limited to) groups, Abelian groups, lattices, monoids, modules (over a fixed ring), algebras (over a fixed ring), Boolean algebras, and Heyting algebras.

DEFINITION 3. Let $\mathcal{V}$ be a variety and let X be some new elements. The [free \$\mathcal{V}\$ -algebra on \$X\$](#) is the set of terms coming from the signature of $\mathcal{V}$ with variables from X , modulo the relation where two terms are equivalent if every object in $\mathcal{V}$ interprets them to be equal. We denote the free $\mathcal{V}$ -algebra on n objects $\{x_1, \dots, x_n\}$ by $\mathbb{F}_{\mathcal{V}}(n)$.

DEFINITION 4. Let C be an object in a category $\mathcal{C}$. We say C is a **generator** for $\mathcal{C}$ if the functor $\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \mathcal{S}et$ is faithful. We say C is an **extremal generator** for $\mathcal{C}$ if $\text{Hom}_{\mathcal{C}}(C, -)$ is faithful and map f is an isomorphism whenever $\text{Hom}_{\mathcal{C}}(C, f)$ is an isomorphism.

EXAMPLE 5. For any variety $\mathcal{V}$, the free $\mathcal{V}$ -algebra $\mathbb{F}_{\mathcal{V}}(1) \in \mathcal{V}$ on one element is an extremal generator for $\mathcal{V}$.

DEFINITION 6. If $\mathcal{C}$ is a category and $U : \mathcal{C} \rightarrow \mathcal{S}et$ is a faithful functor then $(\mathcal{C}, U)$ is a **concrete category**.

EXAMPLE 7. Let $\mathcal{V}$ be a variety. Let $|-| : \mathcal{V} \rightarrow \mathcal{S}et$ forget the algebraic structure involved. Then $(\mathcal{V}, |-|)$ is a concrete category.

DEFINITION 8. Let $F : (\mathcal{C}, U) \rightarrow (\mathcal{D}, V)$ be a functor between concrete categories. We say F is a **concrete functor** if $V \circ F = U$. We say F is a **concrete equivalence** if it is a concrete functor and it is an equivalence of categories, and in this case we say $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are **concretely equivalent**. We say F is a **concrete isomorphism** if it is a concrete functor and it is an isomorphism of categories, and in this case we say $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are **concretely isomorphic**.

REMARK 9. While the inverse of a concrete isomorphism will again be a concrete isomorphism, the equivalence inverses of a concrete equivalence will not necessarily be concrete.

With the language developed above, we wish to determine all varieties equivalent to a given variety $\mathcal{V}$, up to *concrete* isomorphism.

Lawvere Theories

DEFINITION 10. A **Lawvere theory** is a category with countably many objects $\{C_n; n \in \omega\}$ together with distinguished cones, $(\pi_i^n : C_n \rightarrow C_1; 1 \leq i \leq n)$ for each $n \in \omega$, which are products. A **morphism of Lawvere theories** is a morphism of categories which preserves the distinguished cones.

DEFINITION 11. Let $\mathcal{V}$ be a variety. For $n \in \omega$ and $1 \leq i \leq n$, let π_i^n be the homomorphic extension of $x_1 \mapsto x_i : \mathbb{F}_{\mathcal{V}}(1) \rightarrow \mathbb{F}_{\mathcal{V}}(n)$. Consider the full category of $\mathcal{V}$ formed by the free $\mathcal{V}$ -algebras $\{\mathbb{F}_{\mathcal{V}}(n); n \in \omega\}$ together with the cocones $(\pi_i^n : \mathbb{F}_{\mathcal{V}}(1) \rightarrow \mathbb{F}_{\mathcal{V}}(n); 1 \leq i \leq n)$ for each $n \in \omega$. The **theory of $\mathcal{V}$** , denoted $\text{Th}(\mathcal{V})$, is the dual of the above.

EXAMPLE 12. The theory of a variety is a Lawvere theory. Indeed, the free algebra on n objects is the n -fold product of the free algebra on 1 object in this way.

DEFINITION 13. Let $\mathbb{T}$ be a Lawvere theory. The [category of \$\mathbb{T}\$ -models](#), denoted $\text{Mod}(\mathbb{T})$, consists of all functors $\mathbb{T} \rightarrow \mathcal{S}et$ which preserve finite products, as a full subcategory of $\mathcal{S}et^{\mathbb{T}}$.

DEFINITION 14. Let $\mathbb{T}$ be a Lawvere theory with objects $\{C_n; n \in \omega\}$. The [evaluation at \$C_1\$ functor](#), $U^{\mathbb{T}} : \text{Mod}(\mathbb{T}) \rightarrow \mathcal{S}et$, is defined as follows. For objects F, G of $\text{Mod}(\mathbb{T})$ and a natural transformation $f : F \rightarrow G$, take $U^{\mathbb{T}}(F) := F(C_1)$ and $U^{\mathbb{T}}(f) := f_{C_1}$.

EXAMPLE 15. For a Lawvere theory $\mathbb{T}$, $(\text{Mod}(\mathbb{T}), U^{\mathbb{T}})$ is a concrete category.

PROPOSITION 16. Let $\mathcal{V}$ be a variety. Then $(\mathcal{V}, |-|)$ and $(\text{Mod}(\text{Th}(\mathcal{V})), U^{\text{Th}(\mathcal{V})})$ are concretely equivalent.

Proof. We will define concrete functors $\tilde{} : (\mathcal{V}, |-|) \rightarrow (\text{Mod}(\text{Th}(\mathcal{V})), U^{\text{Th}(\mathcal{V})})$ and $\Phi : (\text{Mod}(\text{Th}(\mathcal{V})), U^{\text{Th}(\mathcal{V})}) \rightarrow (\mathcal{V}, |-|)$.

We start with $\tilde{}$. Suppose $A, B \in \mathcal{V}$ and $f : A \rightarrow B$. We define $\tilde{A} \in \text{Mod}(\text{Th}(\mathcal{V}))$ to be the functor $\text{Th}\mathcal{V} \rightarrow \mathcal{S}et$ as follows. It maps a free $\mathcal{V}$ -algebra $\mathbb{F}_{\mathcal{V}}(n)$ to $|A|^n$ (recall free $\mathcal{V}$ -algebras were precisely the objects of $\text{Th}\mathcal{V}$, and $|A|$ is the underlying set of A). Now given a $\text{Th}(\mathcal{V})$ -morphism $\sigma : \mathbb{F}_{\mathcal{V}}(n) \rightarrow \mathbb{F}_{\mathcal{V}}(m)$ in $\text{Th}(\mathcal{V})$, it corresponds to a $\mathcal{V}$ -homomorphism $\sigma' : \mathbb{F}_{\mathcal{V}}(m) \rightarrow \mathbb{F}_{\mathcal{V}}(n)$, (since $\text{Th}(\mathcal{V})$ was defined as a dual). Now σ' is determined by the images $\bar{t}_1, \dots, \bar{t}_m \in \mathbb{F}_{\mathcal{V}}(n)$ of the m generators of $\mathbb{F}_{\mathcal{V}}(m)$ (recall the free algebra consists of terms from $\mathcal{V}$ modulo an equivalence). We let $\tilde{A}$ map σ to

$$(a_1, \dots, a_n) \mapsto (t_1^A(a_1, \dots, a_n), \dots, t_m^A(a_1, \dots, a_n)) : |A|^n \rightarrow |A|^m.$$

Note that this is independent of the choice of representative t_i of $\bar{t}_i$ because the equivalence relation was that which equates terms which are equal under any possible interpretation (so in particular, they are equal under the interpretation of A on $(a_1, \dots, a_n)$). Finally for the morphism $f : A \rightarrow B$. For $\mathbb{F}_{\mathcal{V}}(n) \in \text{Th}(\mathcal{V})$ set $\tilde{f}_{\mathbb{F}_{\mathcal{V}}(n)} := f^n : |A|^n \rightarrow |B|^n$.

Then it is routine to check that $\tilde{}$ is a functor, and it is concrete because $U^{\text{Th}(\mathcal{V})}(\tilde{A}) = \tilde{A}(\mathbb{F}_{\mathcal{V}}(1)) = |A|^1 = |A|$ and similarly for morphisms.

Next we define Φ . Suppose $F, G \in \text{Mod}(\text{Th}(\mathcal{V}))$ (so $F, G : \text{Th}(\mathcal{V}) \rightarrow \mathcal{S}et$ preserve finite products) and $f : F \rightarrow G$ is a natural transformation. We define the $\mathcal{V}$ -algebra $\Phi(F)$ first. It will have universe $F(\mathbb{F}_{\mathcal{V}}(1))$. Now as F preserves products, it is straight-forward to show that there is a unique $F' \in \text{Mod}(\text{Th}(\mathcal{V}))$ which is naturally isomorphic to F with the property that for each $n \in \omega$, $F'(\mathbb{F}_{\mathcal{V}}(n)) = (F(\mathbb{F}_{\mathcal{V}}(1)))^n$ (in particular, F' and F agree on $\mathbb{F}_{\mathcal{V}}(1)$). Suppose σ is an n -ary function symbol in the signature of $\mathcal{V}$. Then $\sigma^{\mathbb{F}_{\mathcal{V}}(n)} : \mathbb{F}_{\mathcal{V}}(n) \rightarrow \mathbb{F}_{\mathcal{V}}(1)$, so we let $\sigma^{\Phi(F)} := F'(\sigma^{\mathbb{F}_{\mathcal{V}}(n)}) : F'(\mathbb{F}_{\mathcal{V}}(n)) \rightarrow F'(\mathbb{F}_{\mathcal{V}}(1))$. Indeed, this has the desired domain as $F'(\mathbb{F}_{\mathcal{V}}(n)) = (F(\mathbb{F}_{\mathcal{V}}(1)))^n$. For the natural transformation, we take $\Phi(f) := f_{\mathbb{F}_{\mathcal{V}}(1)}$.

It is routine to check that Φ is a functor, and it is concrete because $|\Phi(F)| = F(\mathbb{F}_{\mathcal{V}}(1)) = U^{\text{Th}(\mathcal{V})}(F)$ and similarly for morphisms.

It remains to check that these witness an equivalence of categories. This really just amounts to chasing the definitions above; there is nothing tricky involved. I won't include it here, as this proof is long enough as it is. ■

The need to bring in F' in place of F for the definition of Φ is more of a technical detail that must be addressed, rather than an important conceptual leap. It's necessity comes from the fact that in a variety, products are unique only up to isomorphism, whereas for an algebra, powers of the algebra have a particular universe (namely the cartesian product of the original universe). We simply introduce F' to ensure these line up as needed.

PROPOSITION 17. *Let $\mathbb{T}$ be a Lawvere theory. Then there exists a variety $\mathcal{V}_{\mathbb{T}}$ such that $(\mathcal{V}_{\mathbb{T}}, |-|)$ and $(\text{Mod}(\mathbb{T}), U^{\mathbb{T}})$ are concretely equivalent.*

Proof. Let $\{C_n; n \in \omega\}$ be the objects of $\mathbb{T}$, and let $(\pi_i^n : C_n \rightarrow C_1; 1 \leq i \leq n)$ for the distinguished cones of $\mathbb{T}$, for each $n \in \omega$. To define $\mathcal{V}_{\mathbb{T}}$, we must construct a signature and a set of identities to be satisfied. First for the signature. The n -ary basic functions will be the $\mathbb{T}$ -morphisms $C_n \rightarrow C_1$. That was easy enough.

Now we create a set of identities. Suppose t is a term coming from this signature and variables $\{x_n; n \geq 1\}$. By induction on the term t , we define an associated term t' . Our identities will consist of all those of the form $t = t'$.

If $t = x_i$ is a variable, then $t' = x_i$ also. Otherwise $t = f(t_1, \dots, t_n)$ for some $n \geq 1$ and some n -ary function symbol f (that is, a $\mathbb{T}$ -morphism $C_n \rightarrow C_1$). If $f = \pi_i^n$, then $t' = t'_i$. Otherwise, $t' = f(t'_1, \dots, t'_n)$.

It remains to check that $(\mathcal{V}_{\mathbb{T}}, |-|)$ and $(\text{Mod}(\mathbb{T}), U^{\mathbb{T}})$ are concretely equivalent. In the interest of space, we omit this. If the previous proof was understood, then it should be straight-forward. ■

DEFINITION 18. For a Lawvere theory $\mathbb{T}$, the above variety $\mathcal{V}_{\mathbb{T}}$ is called the [variety determined by \$\mathbb{T}\$](#) .

As it turns out, the constructions above turn out to be essentially bijective. This is precisely stated as follows.

PROPOSITION 19. *Let $\mathcal{V}$ be a variety and let $\mathbb{T}$ be a Lawvere theory. Then $\mathcal{V}$ is concretely isomorphic to $\mathcal{V}_{\text{Th}(\mathcal{V})}$, and $\mathbb{T}$ is isomorphic to $\text{Th}(\mathcal{V}_{\mathbb{T}})$.*

Now we add some more abstraction to the game. The reason for this is so that we may go back to some of the above results, reinterpret them in this new light, and then attempt to generalize.

DEFINITION 20. Let Q be an object of a class $\mathcal{C}$. Suppose Q has finite copowers. For each $n \in \omega$, consider an n -fold copower nQ of Q . For each $n \in \omega$, take a cocone of injections $(\mu_i^n : Q \rightarrow nQ; 1 \leq i \leq n)$. We define $\text{Th}_{\mathcal{C}}(Q)$ to be the dual of the full subcategory of $\mathcal{C}$ spanned by these objects, together with the corresponding distinguished cones. Note this is a Lawvere theory.

DEFINITION 21. Let Q be an object of a class $\mathcal{C}$. Suppose Q has finite copowers. Consider the functor $x \mapsto \text{Hom}_{\mathcal{C}}(-, x)|_{\text{Th}_{\mathcal{C}}(Q)} : \mathcal{C} \rightarrow \text{Set}^{\text{Th}_{\mathcal{C}}(Q)}$. Since hom-functors preserve finite

products, this functor factors through $\text{Mod}(\text{Th}_{\mathcal{C}}(Q))$. The resulting functor is called the [Yoneda functor](#), denoted $Y_Q : \mathcal{C} \rightarrow \text{Mod}(\text{Th}_{\mathcal{C}}(Q))$.

EXAMPLE 22. Let $\mathcal{V}$ be a variety. Looking back at the definition of $\text{Th}(\mathcal{V})$ and comparing with the definition of $\text{Th}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1))$, we see immediately that they are the same. Moreover $-$ and $Y_{\mathbb{F}_{\mathcal{V}}(1)}$, while not actually equal, are naturally isomorphic. This also follows quite quickly from writing out the definitions.

So we have that for a variety $\mathcal{V}$, $\text{Th}(\mathcal{V}) = \text{Th}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1))$, $\simeq$ and $Y_{\mathbb{F}_{\mathcal{V}}(1)}$ are naturally isomorphic, and $|-|$ and $\text{Hom}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1), -)$ are naturally equivalent (we haven't mentioned this last one yet, but it is immediate). Now one of our propositions above actually says that we have a concrete equivalence

$$Y_{\mathbb{F}_{\mathcal{V}}(1)} : (\mathcal{V}, \text{Hom}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1), -)) \rightarrow (\text{Mod}(\text{Th}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1))), U^{\text{Th}_{\mathcal{V}}(\mathbb{F}_{\mathcal{V}}(1))}).$$

Recall also that $\mathbb{F}_{\mathcal{V}}(1)$ is an extremal generator for $\mathcal{V}$. A natural question now arises: For which objects Q of $\mathcal{V}$ does the above hold (replacing instances of $\mathbb{F}_{\mathcal{V}}(1)$ with Q)?

DEFINITION 23. Let Q be an extremal generator of a category $\mathcal{C}$. We say Q is a [varietal generator](#) for $\mathcal{C}$ if it is projective and finitely presentable.

EXAMPLE 24. Let $\mathcal{V}$ be a variety and let $n \geq 1$. Then the free $\mathcal{V}$ -algebra $\mathbb{F}_{\mathcal{V}}(n)$ is a varietal generator.

The varietal generators of the category of groups are *precisely* the finitely-generated free groups. A similar statement holds for Abelian groups.

There is a notion of compactness of an object in a category, and there are relations between compact objects and varietal generators. It's not terribly involved, but we wouldn't have time to make use of them, and this paper has enough definitions as it is.

Morita Equivalence

DEFINITION 25. We say two Lawvere theories $\mathbb{T}$ and $\mathbb{S}$ are [Morita equivalent](#) if $\text{Mod}(\mathbb{T})$ and $\text{Mod}(\mathbb{S})$ are equivalent.

What can we say about Morita equivalence so far, in the language of the preceding section? Let $\mathbb{T}$ be a Lawvere theory. Suppose $\mathcal{W}$ is a variety and we have an equivalence $\mathcal{W} \rightarrow \mathcal{V}_{\mathbb{T}}$. It is not hard to check that since $\mathbb{F}_{\mathcal{W}}(1)$ is a varietal generator for $\mathcal{W}$, it's image Q will be a varietal generator for $\mathcal{V}_{\mathbb{T}}$ (indeed, just check that equivalences preserve varietal generators). Moreover, this equivalence induces an isomorphism $\text{Th}(\mathcal{W}) \rightarrow \text{Th}_{\mathcal{V}_{\mathbb{T}}}(Q)$ in the obvious way.

In fact, with a little bit more work in the previous section, we would be able to note that a converse holds. Specifically, if there is a varietal generator Q for $\mathcal{V}_{\mathbb{T}}$ such that $\text{Th}(\mathcal{W})$ is isomorphic to $\text{Th}_{\mathcal{V}_{\mathbb{T}}}(Q)$, then $\mathcal{W}$ is equivalent to $\mathcal{V}_{\mathbb{T}}$.

As such, our quest to characterize the Lawvere theories which are equivalent to $\mathbb{T}$ is equivalent to describing the varietal generators of $\mathcal{V}_{\mathbb{T}}$. It must be noted, however, that there is not a bijective correspondence between Lawvere theories equivalent to $\mathbb{T}$ and varietal generators of $\mathcal{V}_{\mathbb{T}}$. As it turns out, a complete understanding of this correspondence has not been achieved. What is known is the following proposition.

PROPOSITION 26. *Let $\mathcal{V}$ be a variety with varietal generators P and Q . Then $\text{Mod}(\text{Th}_{\mathcal{V}}(P))$ and $\text{Mod}(\text{Th}_{\mathcal{V}}(Q))$ are concretely equivalent if and only if there exists an equivalence $\Psi : \mathcal{V} \rightarrow \mathcal{V}$ which maps P to Q .*

There is so much more work to be done to make significant progress here, but it just won't fit. We will finish by describing two final constructions, and stating a theorem or two about how to describe those Lawvere theories which are Morita equivalent to a given Lawvere theory. There will be no more justifications.

Let $\mathbb{T}$ be a Lawvere theory with objects $\{C_k; k \in \omega\}$ and distinguished cones $(\pi_j^k : C_k \rightarrow C_1; 1 \leq j \leq k)$ for each $k \in \omega$, and let $n \in \omega$. We create a category denoted $\mathbb{T}^{[n]}$ as follows. The objects are

$$\overline{C}_k := \underbrace{C_k \times \cdots \times C_k}_{n \text{ copies}}$$

for each object C_k of $\mathbb{T}$. The morphisms consist of all $\mathbb{T}$ -morphisms between these objects. For $k \in \omega$, the distinguished cone is $(\overline{\pi}_j^k : \overline{C}_k \rightarrow \overline{C}_1; 1 \leq j \leq k)$ where $\overline{\pi}_j^k := (\pi_j^k)^n$. This is a Lawvere theory.

Let $\mathbb{T}$ be a Lawvere theory objects $\{C_k; k \in \omega\}$ and distinguished cones $(\pi_j^k : C_k \rightarrow C_1; 1 \leq j \leq k)$ for each $k \in \omega$, and let $u : C_1 \rightarrow C_1$ be an idempotent $\mathbb{T}$ -morphism. We create a category denoted $\mathbb{T}_u$ as follows. The objects are the objects of $\mathbb{T}$. The morphisms $C_k \rightarrow C_j$ are those $\mathbb{T}$ -morphisms $C_k \rightarrow C_j$ which can be factored as

$$C_k \xrightarrow{u^k} C_k \xrightarrow{f} C_j \xrightarrow{u^j} C_j$$

for some $\mathbb{T}$ -morphism f . (Note that u^k is essentially $(u, \dots, u)$ (k copies), *not* repeated application of u . It is, after all, idempotent!) Furthermore, we take the identity object $C_k \rightarrow C_k$ to be u^k . For each $k \in \omega$, the distinguished cone is $(u \circ \pi_j^k \circ u^k : C_k \rightarrow C_1; 1 \leq j \leq k)$. This is a Lawvere theory.

With a couple more definitions and several intermediate results, we would be able to reach the following theorem.

THEOREM 27. *Let $\mathbb{T}$ be a Lawvere theory. The Lawvere theories which are Morita equivalent to $\mathbb{T}$ are, up to isomorphism, precisely the theories of the form $(\mathbb{T}^{[n]})_u$ for $n \geq 1$ and $u : C_n \rightarrow C_n$ an idempotent invertible $\mathbb{T}$ -morphism.*

The above theorem is closely related to the following theorem.

THEOREM 28. *Let $\mathbb{T}$ be a Lawvere theory. The Lawvere theories which are Morita equivalent to $\mathbb{T}$ are, up to isomorphism, precisely the theories of the form $\text{Th}_{\mathcal{V}_{\mathbb{T}}}(Q)$ for varietal generators Q of $\mathcal{V}_{\mathbb{T}}$.*

Having gotten to this point, it is quite unsatisfactory to stop. We encourage the reader to look through [1] for further details, and to complete the story we set out to tell.

References

- [1] H.-E. Porst, *Equivalence for varieties in general and for BOOL in particular*, Algebra Universalis 43 (2-3) (2000) 157-186
- [2] F. Borceux, *Handbook of Categorical Algebra*, Vol 2. Cambridge University Press, 1994, Cambridge.