

Peano Arithmetic and Π_1^0 -Classes

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- Recall that given a language $\mathcal{L}$, a *sentence* is a formula without free variables.
- A set T of sentences is a *theory* if it is consistent and closed under logical consequence. Precisely, there is no sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$, and whenever $T \vdash \psi$ we have $\psi \in T$.
- A theory T is *complete* if for all sentences ϕ we have either $T \vdash \phi$ or $T \vdash \neg\phi$.
- A theory T is (*computably*) *axiomatizable* if there is a computable set of sentences S such that T is the logical closure of S . This means $T = \{\psi; S \vdash \psi\}$.

- We will assume that everything we deal with is computable. That is, everything is countable, and all functions, relations, and constants are effectively given.
- We also assume that languages come equipped with a *Gödel numbering*. That is, an effective way of coding formulas and sentences. So a set of formulas will be identified with its corresponding set of Gödel numbers.

Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ be the *language of arithmetic*. The *theory of Peano arithmetic* (PA) is the logical closure of the following sentences:

- $\forall x \neg x + 1 = 0$
- $\forall x x + 0 = x$
- $\forall x 0 \cdot x = 0$
- $\forall x \forall y x + 1 = y + 1 \rightarrow x = y$
- $\forall x \forall y x + (y + 1) = (x + y) + 1$
- $\forall x \forall y x \cdot (y + 1) = x \cdot y + x$
- $\forall x \forall y x < y + 1 \leftrightarrow (x < y \vee x = y)$
- $\forall x \neg x < 0$
- $\forall x \forall y x < y \vee x = y \vee y < x$
- and for all formulas $\phi(x, y_1, \dots, y_n)$,
$$\forall y_1 \cdots \forall y_n (\phi(0, \bar{y}) \wedge \forall x (\phi(x, \bar{y}) \rightarrow \phi(x + 1, \bar{y}))) \rightarrow \forall x \phi(x, \bar{y})$$

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Corollary

There is no computable complete consistent extension of Peano arithmetic: $\mathbf{0} \notin \mathcal{D}_{PA}$.

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Proof (Sketch).

Recall we assume we are given a Gödel numbering G from ω to the sentences over arithmetic. We identify a finite binary string σ with the sentence

$$\theta(\sigma) = \left(\bigwedge_{\sigma(i)=1} G(i) \right) \wedge \left(\bigwedge_{\sigma(j)=0} \neg G(j) \right).$$

We say a sentence θ “appears to be consistent at stage t ” if there is no derivation of $\neg\theta$ from the first t axioms of PA in fewer than t lines. Then the relation $R(\sigma, t)$ given by “ $\theta(\sigma)$ appears to be consistent at stage t ,” is computable.

Proof (Sketch (continued)).

Since $R(\sigma, t)$ is computable, the class

$$\{f \in 2^\omega; (\forall n)(\forall t < n)R(f \upharpoonright t, n)\}$$

is a Π_1^0 class. But f is a member of this class if and only if the corresponding set of sentences $G(f^{-1}(\{1\}))$ is a complete consistent extension of PA. □

- Note that we did not use anything particular to Peano arithmetic.
- In fact, the theorem applies to any computably axiomatizable theory in a computable setting, with the same proof.

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Theorem (Gödel's Incompleteness Theorem, Effective Version)

From a description of a consistent, computably axiomatizable theory T which extends PA, we can effectively find a sentence which is independent of T . Such a sentence is called a Gödel-Rosser sentence of T .

Definition

Let $f : \omega \rightarrow \omega$ be a function. We say f is *diagonally noncomputable* (*d.n.c.*) if for all $e \in \omega$, if $\varphi_e(e) \downarrow$ then $f(e) \neq \varphi_e(e)$.

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Let $f : \omega \rightarrow \omega$ be a function and let $n \in \omega$. We say f is *n-valued* if for all $e \in \omega$, $f(e) < n$.

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(i) $\implies$ (ii).

This is immediate. $\square$

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(ii) $\implies$ (iii).

Let T be a complete consistent extension of PA with $\deg(T) = \mathbf{c}$. Let g be the partially computable function $g(e) = \varphi_e(e)$. By other results, there is a formula $\psi(x, y)$ which represents g . That is,

$$\begin{aligned} g(x) \downarrow = y &\Leftrightarrow PA \vdash \psi(x, y), \\ g(x) \downarrow \neq y &\Leftrightarrow PA \vdash \neg \psi(x, y). \end{aligned}$$

Define

$$f(e) = \begin{cases} 1 & \text{if } \psi(e, 0) \in T, \\ 0 & \text{if } \neg \psi(e, 0) \in T. \end{cases}$$

Then f is a **d**-computable 2-valued d.n.c function. □

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(iii) $\implies$ (iv).

Let f be a 2-valued d.n.c. **d**-computable function. Suppose g is a partially computable 2-valued function. Consider the partially computable function

$$\psi(x, y) = g(x).$$

By the s.n.m theorem, there is a computable function h such that

$$\varphi_{h(x)}(y) = g(x).$$

Then

$$1 - f \circ h$$

is a total **d**-computable 2-valued extension of g . □

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(iv) $\implies$ (v).

Suppose $\mathcal{C}$ is a non-empty Π_1^0 class. Let T be a computable tree with $\mathcal{C} = [T]$. Let $h : \omega \rightarrow 2^{<\omega}$ be a computable bijection. Define the partially computable function

$$f(e) = \begin{cases} i & \text{if } i \in \{0, 1\}, h(e) \in T, \text{ and there exists a level } \\ & l \text{ such that } h(e)^{\frown} i \text{ has a descendent at level } l \text{ in} \\ & T, \text{ but } h(e)^{\frown} (1 - i) \text{ does not} \\ \uparrow & \text{otherwise.} \end{cases}$$

As f is 2-valued, it has a **d**-computable 2-valued extension g .

(iv) $\implies$ (v) (continued).

We construct a **d**-computable infinite path through T by stages.

Let σ_0 be the empty string. Note $\sigma_0 \in T^{\text{ext}}$.

Stage $s + 1$: Suppose $\sigma_s \in T^{\text{ext}}$. Define

$$\sigma_{s+1} = \sigma_s \hat{\ } (g \circ h^{-1})(\sigma_s).$$

Then $\sigma_{s+1} \in T^{\text{ext}}$. This completes the construction.

Now $\sigma = \bigcup_s \sigma_s \in [T^{\text{ext}}] = [T] = \mathcal{C}$ is **d**-computable, so take $\mathbf{c} = \text{deg}(\sigma)$. □

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(v) $\implies$ (vi).

Suppose A, B is a computably inseparable pair of c.e. sets. Then the class $\mathcal{C}$ of separating sets for A, B is a Π_1^0 class. Moreover, it is non-empty, so there is a member $C \in \mathcal{C}$ which is **d**-computable. That is, C is a separating set of A, B with $C \leq \mathbf{d}$. □

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(vi) $\implies$ (i).

Fix any order of sentences and any order of generating proofs. Let A denote the set of pairs (F, ψ) such that:

- $F \cup \{\psi\}$ is a finite set of sentences;
- A proof of a contradiction is found from $PA \cup F \cup \{\psi\}$;
- No proof of a contradiction is found from $PA \cup F \cup \{\neg\psi\}$ before this point.

Let B denote the set of pairs (F, ψ) such that:

- $F \cup \{\psi\}$ is a finite set of sentences;
- A proof of a contradiction is found from $PA \cup F \cup \{\neg\psi\}$;
- No proof of a contradiction is found from $PA \cup F \cup \{\psi\}$ before this point.

Then A, B are disjoint c.e. sets. So let C be a $\mathbf{d}$ -computable separating set of A, B . Also let $D \in \mathbf{d}$.

$(vi) \implies (i)$ (continued).

We construct a complete consistent degree $\mathbf{d}$ extension T of PA, as well as a bijection g from ω to sentences, in stages. At stage $n + 1$, we will define $g(n)$ and decide whether or not $g(n)$ is a member of T .

Construction:

Stage 0: Let $T_0 = \emptyset$, $V_0 = \emptyset$, and $U_0 = \emptyset$. Note $\text{PA} \cup T_0$ is consistent.

Stage $n + 1$: Suppose $\text{PA} \cup T_n$ is consistent. First we define $g(n)$. If n is even, let $g(n)$ be the Gödel-Rosser sentence of the logical closure of $\text{PA} \cup T_n$. If n is odd, let $g(n)$ be the first sentence not yet in the range of g .

(vi) $\implies$ (i) (continued (continued)).

Next we decide whether $g(n)$ is a member of T . If n is even, then enumerate $g(n)$ into V_{n+1} if and only if $\frac{n}{2} \in D$ (and into U_{n+1} otherwise). If n is odd, then enumerate $g(n)$ into V_{n+1} if and only if $(T_n, g(n)) \notin C$ (and into U_{n+1} otherwise).

Now define

$$T_{n+1} = \begin{cases} T_n \cup \{g(n)\} & \text{if } g(n) \in V_{n+1}, \\ T_n \cup \{\neg g(n)\} & \text{if } g(n) \in U_{n+1}. \end{cases}$$

Then $\text{PA} \cup T_{n+1}$ is consistent.

Finally, let

$$T = \text{PA} \cup \left(\bigcup_n T_n \right).$$

This completes the construction.

(vi) $\implies$ (i) (continued (continued (continued))).

Note:

- T is an extension of PA;
- T is consistent;
- T is complete.

Therefore T is a complete consistent extension of PA, and so $\deg(T)$ is a PA degree. But $\deg(T) = \mathbf{d}$. Thus $\mathbf{d} \in \mathcal{D}_{PA}$ as desired. □