easier to state and more true

2015 02 01

Model Theory

Def A *structure* $\mathcal{M}$ consists of

- a non-empty set M, *universe* or *underlying set* of $\mathcal{M}$
- a sequence $(c_i;\ i \in I_{con})$ of distinguished elements of M, called *constants* of $\mathcal{M}$.
- a sequence $(f_i : M^{n_i} \to M;\ i \in I_{fun})$ of M-valued functions on certain finite cartesian powers of M, called the *basic functions* of $\mathcal{M}$
- a sequence of subsets $(R_i \subseteq M^{k_i};\ i \in I_{rel})$ of finite cartesian powers of M, called the *basic relations* of $\mathcal{M}$

signature of $\mathcal{M}$

Remark: A 0-ary function $f : M^0 \to M$, $M^0 = \{\emptyset\} = 1$. So f picks out a single element of M. So by including such in the constants of $\mathcal{M}$ we can (and do) assume that all the basic functions are of arity greater than 0.

Example: Consider $\mathbb{R}$. Deciding what structure to put on $\mathbb{R}$ corresponds to what aspects of $\mathbb{R}$ you wish to study.

- $\mathbb{R}$ as a pure set: -universe $\mathbb{R}$, empty signature $(\mathbb{R})$
- $\mathbb{R}$ as a linear order: -universe $\mathbb{R}$, one relation $< \subseteq \mathbb{R}^2$ $(\mathbb{R}, <)$
- additive groups of reals: $\mathcal{M} = (\mathbb{R}, \underset{\text{constant}}{0}, \underset{\text{binary func.}}{+}, \underset{\text{unary func.}}{-})$
- ring of real numbers:
  $\mathcal{M} = (\mathbb{R}, 0, 1, +, -, \times)$
- ordered ring $\mathcal{M} = (\mathbb{R}, 0, 1, +, -, \times, <)$
- real exp $\mathcal{M} = (\mathbb{R}, 0, 1, +, -, \times, <, \exp)$

Def Suppose $\mathcal{M}$ and $\mathcal{N}$ are structures. We say that $\mathcal{N}$ is an *expansion* of $\mathcal{M}$, or $\mathcal{M}$ is a *reduct* of $\mathcal{N}$ if they have the same universe and the signature of $\mathcal{M}$ is contained in the signature of $\mathcal{N}$.

Ex $(\mathbb{C}, 0, 1, +, -, \times)$
$(\mathbb{Z}/3\mathbb{Z}, 0, 1, +, -, \times)$

<u>Def</u> A <u>language</u> L consists of three sets of symbols:

- L^{con} a set of <u>constant symbols</u>
- L^{fun} a set of <u>function symbols</u>, together with a positive integer n_f for each $f \in L^{fun}$, called the <u>arity</u> of f
- L^{rel} set of <u>relation symbols</u>, together with a natural number k_R for each $R \in L^{rel}$, called the <u>arity</u> of R

An <u>L-structure</u> is a structure $\mathcal{M}$ together with a bijective correspondence between L and the signature of M:

$$L^{con} \longleftrightarrow I^{con}$$
$$L^{fun} \longleftrightarrow I^{fun}$$
$$L^{rel} \longleftrightarrow I^{rel}$$

preserving arity. That is, to each constant symbol $c \in L^{con}$ is associated a constant of $\mathcal{M}$, $c^{\mathcal{M}}$, to each n-ary function symbol $f \in L^{fun}$ is associated an n-ary basic function $f^{\mathcal{M}}$ of $\mathcal{M}$, and to each k-ary relation symbol $R \in L^{rel}$ is associated a k-ary basic relation $R^{\mathcal{M}}$ of $\mathcal{M}$. We call these $c^{\mathcal{M}}, f^{\mathcal{M}}, R^{\mathcal{M}}$ the <u>interpretation</u> of c, f, R in $\mathcal{M}$.

Common abuse of notation: We do not always distinguish notationally between the symbol in L and its interpretation in $\mathcal{M}$.

$L = \{0, 1, +, -, \times\}$ is called the <u>language of rings</u>. Every ring is naturally an L-structure. <u>Not</u> all L-structures are rings!

ex $\mathcal{M} = (\mathbb{R}, 0^{\mathcal{M}} = \pi, 1^{\mathcal{M}} = -2, +^{\mathcal{M}}: (a,b) \mapsto 0, -^{\mathcal{M}}: a \mapsto a, \times^{\mathcal{M}}: (a,b) \mapsto a+b$

<u>Def</u> Suppose L is a language and $\mathcal{M}$ and $\mathcal{N}$ are L-structures. Then an <u>L-embedding</u> $j: \mathcal{M} \to \mathcal{N}$ is an injective function $j: M \to N$ satisfying

(1) for all constant symbols $j(c^{\mathcal{M}}) = c^{\mathcal{N}}$

(2) for all n-ary $f \in L^{fun}$, $a_1, \dots, a_n \in M$,
$$j(f^{\mathcal{M}}(a_1, \dots, a_n)) = f^{\mathcal{N}}(j(a_1), \dots, j(a_n)).$$

(3) for all k-ary $R \in L^{rel}$, $a_1, \dots, a_k \in M$,
$$(a_1, \dots, a_k) \in R^{\mathcal{M}} \iff (j(a_1), \dots, j(a_k)) \in R^{\mathcal{N}}.$$

In the case when $M \subseteq N$ and the containment map

$$j: M \to N$$
$$a \mapsto a$$

is an L-embedding, we say that $\mathcal{M}$ is an <u>L-substructure</u> of $\mathcal{N}$. This is denoted $\mathcal{M} \subseteq \mathcal{N}$.

Exercises:

(a) $\mathcal{M} \subseteq \mathcal{N} \iff M \subseteq N,$

$$c^{\mathcal{M}} = c^{\mathcal{N}} \quad \text{for all } c \in L^{con}$$
$$f^{\mathcal{M}} = f^{\mathcal{N}}|_{M^n} \quad \text{for all } n\text{-ary } f \in L^{fun}$$
$$R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^k \quad \text{for all } k\text{-ary } R \in L^{rel}$$

(b) Suppose $\mathcal{N}$ L-structure, $M \subseteq N$. M is the universe of a (uniquely determined) substructure $\mathcal{M} \subseteq \mathcal{N}$ iff

- M contains all constants of $\mathcal{N}$
- M is closed under all basic functions of $\mathcal{N}$
- $M \neq \emptyset$

(c) Suppose $j: \mathcal{M} \to \mathcal{N}$ is an L-embedding. Let $M' := j(M)$. Then M' is the universe of a L-substructure $\mathcal{M}'$ of $\mathcal{N}$, and $j: \mathcal{M} \to \mathcal{M}'$ is an L-isomorphism.

(An L-isomorphism is a surjective L-embedding.)

So instead of $j: \mathcal{M} \to \mathcal{N}$, we may as well consider $\mathcal{M}' \subseteq \mathcal{N}$.

Choosing a language determines the substructures.

structure	substructures
$(\mathbb{R})$	non-empty subsets
$(\mathbb{R}, <)$	"
$(\mathbb{R}, 0, +, -)$	subgroups
$(\mathbb{R}, 0, 1, +, -, \times)$	subrings

<u>Examples</u>: Fix a field F.

$$L = \{0, +, -, \underset{\substack{\uparrow \\ \text{unary func symbol}}}{(\lambda_a)_{a \in F}}\}$$

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is the language of F-vector spaces. Every F-vector space is naturally an L-structure $\mathcal{V}$:

$0^{\mathcal{V}}$ zero vector
$+^{\mathcal{V}}$ vector addition
$-^{\mathcal{V}}$ negative of a vector
$\lambda_a^{\mathcal{V}}$ scalar multiplication by a $V \to V : v \mapsto av$.

The substructures of $\mathcal{V}$ are F-subspaces.

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Terms. Fix a language L.

Var = $\{x_0, x_1, \dots\}$ infinite countable set of symbols called variables symbols.

Def| The set of L-terms is the smallest set of finite strings of symbols from:

- L
- Var
- (
-)
- ,

satisfying:

(i) every constant symbol in L is an L-term
(ii) every variable is an L-term
(iii) if $f \in L^{\text{fun}}$ is n-ary, and $t_1, \dots, t_n$ are L-terms then $f(t_1, \dots, t_n)$ is an L-term.

We write $t = t(x_1, \dots, x_n)$ to mean that the variables appearing in t come from $\{x_1, \dots, x_n\}$.

Abuse of notation: We often write terms in an informal but more readable fashion.

Ex $L = \{0, 1, +, -, \times\}$
Instead of $\times(+(x_0, -(x_1)), \times(1, x_2))$ we write $(x_0 + (-x_1))(1x_2)$

Note: We do not simplify $1x_2$ by x_2 as this is not always "true" in every L-structure.

Interpreting Term

$\mathcal{M}$ an L-structure with universe M, $t = t(x_1, \dots, x_n)$ term
We define the <u>interpretation</u> of t in $\mathcal{M}$ to be

$$t^{\mathcal{M}} : M^n \longrightarrow M$$

define recursively as follows:

(i) if t is $c \in L^{con}$ then

$$(a_1, \dots, a_n) \longmapsto c^{\mathcal{M}}$$

(ii) if $t = x_i$ then

$$(a_1, \dots, a_n) \longmapsto a_i$$

(iii) if $t = f(t_1, \dots, t_m)$, $f \in L^{fun}$ m-ary, $t_i = t_i(x_1, \dots, x_n)$

$$(a_1, \dots, a_n) \longmapsto f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \dots, a_n), \dots, t_m^{\mathcal{M}}(a_1, \dots, a_n))$$

Remark: $t^{\mathcal{M}}$ depends not only on t but on the representation $t = t(x_1, \dots, x_n)$.

Example: $t = x$ variable. $\mathcal{M}$ L-structure. If we write $t = t(x)$ then

$$t^{\mathcal{M}} : M \to M$$

identity. If we write $t = t(x,y)$,

$$t^{\mathcal{M}} : M^2 \to M$$
$$(a,b) \longmapsto a$$

If we write $t = t(y,x)$,

$$t^{\mathcal{M}} : M^2 \to M$$
$$(a,b) \longmapsto b$$

M non-empty?

Exercise: $\mathcal{N}$ an L-structure, $M \subseteq N$. Then M is the universe of a (unique) substructure of $\mathcal{N}$ if and only if M is closed under $t^{\mathcal{N}}$ for all L-terms t.

Remark: $\mathcal{T} := \{t^{\mathcal{M}} ; t \text{ L-terms}\}$, $\mathcal{M}$ L-structure
is the smallest set of M-valued functions on (various) cartesian powers of M that ~~is closed under~~: satisfies:

- contains all constant functions $c^{\mathcal{M}}$, $c \in L^{con}$
- contains all coordinate projections
- contains all basic functions of $\mathcal{M}$

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<u>Def</u>] An <u>atomic L-formula</u> is a finite string of symbols from L; (;) ; , ; Var ; = , of the form

(i) $(t = s)$ where t, s are L-terms

(ii) $R(t_1, \dots, t_k)$ where $R \in L^{rel}$ is k-ary, $t_1, \dots, t_k$ are L-terms.

For readability, we write $x_0 < x_1^2$ rather than the more correct $<(x_0, \times(x_1, x_1))$

$L = \{0, 1, +, -, \times, <$

<u>Def</u>] The set of <u>L-formulas</u> is the smallest set of finite strings of symbols from L; (;) ; , ; Var ; = ; $\wedge$; $\vee$; $\neg$; $\forall$; $\exists$, satisfying

(i) every atomic L-formula is an ~~atomic~~ L-formula

(ii) if ϕ, ψ are L-formulas then so are $(\phi \vee \psi)$, $\neg\phi$, $(\phi \wedge \psi)$

(iii) if ϕ is an L-formula, and $x \in Var$, then $\exists x \phi$ and $\forall x \phi$ are L-formulas.

Abbreviations: We write

$(\phi \to \psi)$ for $(\neg \phi \vee \psi)$

$(\phi \leftrightarrow \psi)$ for $(\phi \to \psi) \wedge (\psi \to \phi)$

<u>Def</u>] ϕ an L-formula. An occurence of a variable x in ϕ is called <u>bound</u> if it appears in the scope of a quantifier $(\forall, \exists)$. Otherwise, the variable occurence is said to be <u>free</u>. An <u>L-sentence</u> is an L-formula where all variables occur bound.

<u>Ex</u> $L = \{\in\}$

$$(x \in y) \wedge (\forall z (z \in x \to z \in y)) \wedge (\forall z ((z \in y) \to (z \in x) \vee (z = x)))$$

free free bound free bound free

This formula "says" something about x, y, namely $y = S(x)$.

We write $\phi = \phi(x_1, \dots, x_n)$ to mean that the free variables in ϕ are from $\{x_1, \dots, x_n\}$.

<u>Ex</u> $(x < 0) \vee (\exists x (x^2 = 1))$ $\qquad$ $(x < 0) \vee (\exists z (z^2 = 1))$

free bound

For simplicity, we just assume that no variable occurs free and bound in the same formula.

Example: language of $\mathbb{C}$-vector spaces.

$$L = \{0, +, -, (\lambda_c)_{c \in \mathbb{C}}\}$$

$$?? \quad \lambda_{2i}(x+y) = \lambda_{2i}(x) + \lambda_{2i}(y) \qquad \phi(x,y)$$

$$\sigma = \forall x \forall y \, \phi(x,y)$$

L-sentence

What it means for σ to be true in an L-structure $\mathcal{M}$.

It should mean that whenever $v, u \in M$, (v,u) satisfy $\phi(x,y)$.

In this case, satisfy should mean

$$\lambda_{2i}^{\mathcal{M}}\left(+^{\mathcal{M}}(v,u)\right) = +^{\mathcal{M}}\left(\lambda_{2i}^{\mathcal{M}}(v), \lambda_{2i}^{\mathcal{M}}(u)\right).$$

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Tarski's definition of truth

Def) $\mathcal{M}$ L-structure, universe M, $\phi(x_1, .., x_n)$ an L-formula

$a = (a_1, \ldots, a_n) \in M^n \qquad x = (x_1, \ldots, x_n)$

We define what it means for a to realise ϕ in $\mathcal{M}$, or $\phi(a)$ is true in $\mathcal{M}$, or a satisfies ϕ in $\mathcal{M}$, denoted $\mathcal{M} \models \phi(a)$, by recursion on complexity of ϕ:

atomic

(i) ϕ is $(t_1 = t_2)$ where $t_1(x_1, .., x_n)$ and $t_2(x_1, .., x_n)$ are L-terms:

$\mathcal{M} \models \phi(a)$ iff $t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a)$.

(ii) ϕ is $R(t_1, .., t_k)$, $R \in L^{rel}$ k-ary, $t_1, .., t_k$ are L-terms

$\mathcal{M} \models \phi(a)$ iff $(t_1^{\mathcal{M}}(a), \ldots, t_k^{\mathcal{M}}(a)) \in R^{\mathcal{M}}$

(iii) if ϕ is $\neg\psi$ then

$\mathcal{M} \models \phi(a)$ iff $\mathcal{M} \not\models \psi(a)$

(iv) $\phi = \psi_1 \wedge \psi_2$ then

$\mathcal{M} \models \phi(a)$ iff $\mathcal{M} \models \psi_1(a)$ and $\mathcal{M} \models \psi_2(a)$

(v) $\phi = \psi_1 \vee \psi_2$ then

$\mathcal{M} \models \phi(a)$ iff $\mathcal{M} \models \psi_1(a)$ or $\mathcal{M} \models \psi_2(a)$

(vi) $\phi(x) = \exists z \psi$. Note $\psi = \psi(x, z)$

$\mathcal{M} \models \phi(a)$ iff there is $b \in M$ s.t. $\mathcal{M} \models \psi(a, b)$

(vii) $\phi = \forall z \psi$ Note $\psi = \psi(x, z)$

$\mathcal{M} \models \phi(a)$ iff for all $b \in M$, $\mathcal{M} \models \psi(a, b)$

The set

$$\{a \in M^n ; \mathcal{M} \models \phi(a)\} \subseteq M^n$$

is denoted by $\phi^{\mathcal{M}}$ and is called the <u>set defined by ϕ in $\mathcal{M}$</u>.

What happens if $n=0$. So ϕ is a sentence. $M^0 = 1$ is a singleton. So either the unique element of M^0 satisfies ϕ or not. We denote this by $\mathcal{M} \models \phi$ (ϕ is true in $\mathcal{M}$) or $\mathcal{M} \models \neg\phi$ (ϕ is false in $\mathcal{M}$).

Example: Consider

$\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \times)$

$\phi(x) : \exists z(z^2 = x)$

$\mathcal{R} \models \neg\phi(-1) \quad \mathcal{R} \models \phi(2)$

$\phi^{\mathcal{R}} = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$

$\mathcal{Q} = (\mathbb{Q}, 0, 1, +, -, \times)$

$\mathcal{R} \models \neg\forall x \phi(x)$

$\mathcal{Q} \models \neg\forall x \phi(x)$

$\mathcal{Q} \models \neg\phi(2)$

$\mathcal{C} = (\mathbb{C}, 0, 1, +, -, \times)$

$\mathcal{C} \models \forall x \phi(x)$ ie $\phi^{\mathcal{C}} = \mathbb{C}$

$\psi(x,y) : \exists z(z^2 = y - x)$

$\psi^{\mathcal{R}} = \{(a,b) \in \mathbb{R}^2 ; b \geq a\}$

ie ψ defines $\leq$ on $\mathbb{R}$ in $\mathcal{R}$.

Consider $\mathcal{R} = (\mathbb{R}, <)$.

$\phi(x) : x = x$ then $\phi^{\mathcal{R}} = \mathbb{R}$

$\phi(x) : x \neq x$ then $\phi^{\mathcal{R}} = \emptyset$

What about $(0,1)$? Problem: refers to elements in $\mathcal{R}$.

$(x > 0) \wedge (x < 1)$ is <u>not</u> an $\{<\}$-formula

<u>Def</u> L language, $\mathcal{M}$ L-structure, $B \subseteq M$. Let

$$L_B := L \cup \{\underline{b} : b \in B\}$$

where $\underline{b}$ is a new constant symbol. New language, but there is a canonical expansion of $\mathcal{M}$ to an L_B-structure, namely

$\mathcal{M}_B$ = L_B-structure with universe M, all symbols interpreted in $\mathcal{M}$ exactly as they were in $\mathcal{M}$, and $\underline{b}^{\mathcal{M}_B} = b$.

So canonical that we often drop subscript B and think of $\mathcal{M}$ as an L_B-structure.

$(x > \underline{0}) \wedge (x < \underline{1})$ is an $L_{\mathbb{R}}$-formula, it defines $(0,1)$ in $\mathcal{R}_{\mathbb{R}}$.

We also tend to drop the underscores.

Remark: $\mathcal{M}$ L-structure, $B \subseteq M$. $\phi(x_1, \dots, x_n)$ an L_B-formula. Then there exists an L-formula $\psi(x_1, \dots, x_n, y_1, \dots, y_\ell)$ and $b_1, \dots, b_\ell \in B$ such that $\phi(x_1, \dots, x_n) = \psi(x_1, \dots, x_n, \underline{b_1}, \dots, \underline{b_\ell})$. ← ? means what

<u>Def</u>] $\mathcal{M}$ an L-structure, $B \subseteq M$. A set $X \subseteq M^n$ is <u>definable over B in $\mathcal{M}$</u> (or <u>B-definable in $\mathcal{M}$</u>) if $X = \phi^{\mathcal{M}_B}$ for some L_B-formula ϕ.
X is 0-definable if it is $\emptyset$-definable
X is <u>definable</u> if it is M-definable.

Half+ of model theory is the study of definable sets.

<u>Example</u> <u>Rings</u>

$L = \{0, 1, +, -, \times\}$

$\mathcal{R} = (R, 0, 1, +, -, \times)$ commutative and unitary

What are the definable sets in $\mathcal{R}$?

Let $P_1, \dots, P_\ell \in R[X_1, \dots, X_n]$. By the <u>zero-set</u> of $P_1, \dots, P_n$ we mean

$$V(P_1, \dots, P_\ell) = \{a \in R^n ; P_i(a) = 0 \quad i \in \{1, \dots, \ell\}\}.$$

Such sets are called <u>Zariski closed</u> subsets of R^n, or <u>algebraic subsets</u> of R^n. Zariski closed sets are definable, by

$$\bigwedge_{i=1}^{\ell} \Big(\underbrace{P_i(x_1, \dots, x_n)}_{\text{is an } L_R\text{-term}} = 0 \Big)$$

over the coefficients of the P_i's. Every quantifier-free definable set in $\mathcal{R}$ is a finite boolean comb. of Zariski closed sets.

A3 says that ~~the~~ all L-terms are (in $\mathcal{R}$) of the form $p(x_1, \dots, x_n)$ where $P \in \mathbb{Z}[X_1, \dots, X_n]$, $x = (x_1, \dots, x_n)$. Suppose $\phi(x)$ is $(t(x) = s(x))$ atomic, t, s L_R-terms. Write

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$$t(x) = t'(x, \underline{b})$$
$$s(x) = s'(x, \underline{c})$$

where t', s' are L-terms, and b, c are k-tuples from R. By exercise, we have $P_{s'}, P_{t'} \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_k]$

$$t'^R = P_{t'}, \quad s'^R = P_{s'}$$
$$P_s = P_{s'}(X_1, \ldots, X_n, b_1, \ldots, b_k) \in R[X_1, \ldots, X_n]$$
$$P_t = P_{t'}(X_1, \ldots, X_n, c_1, \ldots, c_k) \in \quad ''$$
$$\Phi(x) \text{ defines } V(\underbrace{P_s - P_t}_{\in R[X_1, X_n]}) \subseteq R^n$$

So the quantifier free ~~sets~~ definable sets will be of the form

$$V_1 \setminus W_1 \cup \cdots \cup V_e \setminus W_e \subseteq R^n$$

V_i's Zariski closed, $W_i \subseteq V_i$ Zariski closed.

Such sets are called Zariski constructable.

Proposition: In a ring, q.f definable = Zariski constructable □

Quantifiers are the complication.

Fact: If R is an algebraically closed fields, then definable = constructable – prove this later.

Fact: If R is a field, characteristic 0. If definable = Zariski-constructable, then R is algebraically closed.

Point: Alg. closed fields are tame.

$R_{\geq 0}$ cannot be defined by q.f. definable set.

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ex $\mathcal{Z} = (\mathbb{Z}, 0, +, \cdot)$, $\mathcal{Q} = (\mathbb{Q}, 0, +, \cdot)$ $\quad \mathcal{Z} \subseteq \mathcal{Q}$

$\psi(x,y): y + y = x$

$a, b \in \mathbb{Z}$, $\mathcal{Z} \models \psi(a,b) \iff 2b = a \iff \mathcal{Q} \models \psi(a,b)$

$\mathcal{Z}$ and $\mathcal{Q}$ agree on integer solutions to $\psi(x,y)$.

Let $\varphi(x): \exists y\, \psi(x,y)$ $\quad \mathcal{Q} \models \varphi(1)$

$\mathcal{Z} \models \neg\varphi(1)$

Proposition: $\mathcal{M} \subseteq \mathcal{N}$ substructure, $\varphi(x)$, $x = (x_1, \dots, x_n)$,
$a = (a_1, \dots, a_n) \in M^n$

(a) If φ is quantifier free, then

$$\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a).$$

(b) If $\varphi(x)$ is $\exists y \psi(x,y)$ where $y = (y_1, \dots, y_m)$, and $\psi(x,y)$ is quantifier free (ie φ is existential)

$$\mathcal{M} \models \varphi(a) \to \mathcal{N} \models \varphi(a)$$

(c) If $\varphi(x)$ is universal,

$$\mathcal{N} \models \varphi(a) \Rightarrow \mathcal{M} \models \varphi(a)$$

Proof: Claim: $t(x)$ an L-term,

$$t^{\mathcal{N}}|_{M^n} = t^{\mathcal{M}}$$

We proceed by induction on complexity of $t(x)$:

- $t = c \in L^{con}$

  $t^{\mathcal{N}}|_{M^n}$ is the constant function on M^n with value $c^{\mathcal{N}}$
- $t(x) = f(t_1(x), \dots, t_m(x))$, $t_1, \dots, t_m$ L-terms, $f \in L^{fn}$ l-ary

  missing variables

  let $a \in M^n$. Then

$$t^{\mathcal{M}}(a) = f^{\mathcal{N}}(t_1^{\mathcal{N}}(a), \dots, t_l^{\mathcal{N}}(a))$$
$$= f^{\mathcal{N}}(t_1^{\mathcal{M}}(a), \dots, t_l^{\mathcal{M}}(a))$$
$$= \quad \ddots$$

We now prove (a) by induction on the complexity of $\varphi(x)$.

- φ is atomic
  - $\varphi(x)$ is $s(x) = t(x)$, s,t L-terms

$$\mathcal{M} \models \varphi(a) \iff s^{\mathcal{M}}(a) = t^{\mathcal{M}}(a)$$
$$\iff s^{\mathcal{N}}(a) = t^{\mathcal{N}}(a)$$
$$\iff \mathcal{N} \models \varphi(a)$$

  - $\varphi(x)$ is $R(t_1, \dots, t_l)$, $t_1, \dots, t_l$ L-terms, $R \in L^{rel}$ l-ary

$$\mathcal{M} \models \varphi(a) \iff (t_1^{\mathcal{M}}(a), \dots, t_l^{\mathcal{M}}(a)) \in R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^l \text{ as substructure}$$
$$\iff (t_1^{\mathcal{M}}(a), \dots, t_l^{\mathcal{M}}(a)) \in R^{\mathcal{N}}$$
$$\iff (t_1^{\mathcal{N}}(a), \dots, t_l^{\mathcal{N}}(a)) \in R^{\mathcal{N}} \text{ by claim}$$
$$\iff \mathcal{N} \models \varphi(a).$$

- $\varphi(x)$ is $\neg\psi(x)$

$$\mathcal{M} \models \varphi(a) \iff \mathcal{M} \not\models \psi(a) \iff \mathcal{N} \not\models \psi(a) \iff \mathcal{N} \models \varphi(a).$$

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- $\varphi(x)$ is $\psi_1(x) \wedge \psi_2(x)$

$$\mathcal{M} \models \varphi(a) \iff \mathcal{M} \models \psi_1(a) \text{ and } \mathcal{M} \models \psi_2(a)$$
$$\iff \mathcal{N} \models \psi_1(a) \text{ and } \mathcal{N} \models \psi_2(a)$$
$$\iff \mathcal{N} \models \varphi(a)$$

- Note $\psi_1 \vee \psi_2$ is equivalent to $\neg(\neg\psi_1 \wedge \neg\psi_2)$, covered by previous inductive cases.

Now we prove (b):

$$\mathcal{M} \models \varphi(a) \iff \text{there is } b \in M^n \text{ st } \mathcal{M} \models \psi(a,b)$$
$$\iff \quad \text{"} \quad \text{"} \quad \mathcal{N} \models \psi(a,b)$$
$$\implies \quad \text{"} \quad b \in N^n \text{ st } \mathcal{N} \models \psi(a,b)$$
$$\iff \mathcal{N} \models \varphi(a)$$

Finally we prove (c): $\varphi(x)$ is $\forall y \psi(x,y)$

Note: $\forall y \psi(x,y)$ is equivalent to $\neg \exists y \neg \psi(x,y)$

Can apply part (b) to $\neg \exists y \neg \psi(x,y)$ contrapositively

$$\mathcal{N} \models \neg \exists y \neg \psi(x,y) \implies \mathcal{M} \models \neg \exists y \neg \psi(x,y)$$
$$\iff \mathcal{M} \models \forall y \psi(x,y)$$

▫ ?

We could have proved the proposition for L-embeddings $j: \mathcal{M} \to \mathcal{N}$:

part (a): $\varphi(x)$ quantifier-free formula, $a \in M^n$

$$\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(ja)$$

where $a = (a_1, \dots, a_n)$, $ja = (j(a_1), \dots, j(a_n))$.

<u>Def</u> An L-embedding $j: \mathcal{M} \to \mathcal{N}$ is <u>elementary</u> if for all L-formulas $\varphi(x)$, $x = (x_1, \dots, x_n)$, all $a \in M^n$,

$$\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(ja).$$

If $\mathcal{M} \subseteq \mathcal{N}$ and the containment map $M \hookrightarrow N$ is elementary, then we say $\mathcal{M}$ is an <u>elementary substructure</u>, denoted by $\mathcal{M} \preccurlyeq \mathcal{N}$

ex $(\mathbb{Z}, 0, +, -) = \mathbb{Z} \not\preccurlyeq \mathbb{Q} = (\mathbb{Q}, 0, +, -)$

since $\varphi(x): \exists y (y + y = x)$, $\mathbb{Q} \models \varphi(1)$, $\mathbb{Z} \not\models \varphi(1)$

Example: $\mathbb{Q}$ has <u>no</u> proper elementary substructures.

let me assume a
is positive, I'm not
sure why

Suppose $G \preceq \mathbb{Q}$.

$\mathbb{Q} \models \exists x (x \neq 0)$ so $G \models \exists x (x \neq 0)$

For each $n > 0$, let σ_n be the sentence

$$\forall x \exists y (\underbrace{y + \cdots + y}_{n \text{ times}} = x)$$

$\mathbb{Q} \models \sigma_n$ for all n, so $G \models \sigma_n$ for all n, so G is divisible
Let $0 \neq \frac{a}{b} \in G$. So $\frac{a}{b} + \cdots + \frac{a}{b} = a \in G$. So $\exists c$ st $ac = a$, so $c = 1$ so $1 \in G$. So $\mathbb{Z} \subseteq G \subseteq \mathbb{Q}$, G divisible $\Rightarrow G = \mathbb{Q}$

Proposition: (Tarski-Vaught test for $\preceq$)
$\mathcal{M} \subseteq \mathcal{N}$, TFAE
(1) $\mathcal{M} \preceq \mathcal{N}$
(2) For every L_M-formula $\varphi(x)$ in one variable, if $\mathcal{N} \models \exists x \varphi(x)$ then there is $a \in M$ such that $\mathcal{N} \models \varphi(a)$.

Proof: (1) $\Rightarrow$ (2): Write $\varphi(x)$ as $\psi(x, \underline{b})$ where $b = (b_1, \ldots, b_m) \in M^m$ and $\psi(x, y)$ is an L-formula, $y = (y_1, \ldots, y_m)$.

$$\mathcal{N} \models \exists x \varphi(x) \Rightarrow \mathcal{N} \models \mathcal{N} \models \exists x \psi(x, b)$$

ie if $\theta(y)$ is $\exists x \psi(x, y)$ then $\mathcal{N} \models \theta(b)$.
By (1), $\mathcal{M} \models \theta(b)$.

$\Rightarrow$ there is an $a \in M$ st $\mathcal{M} \models \psi(a, b) \overset{(1)}{\Rightarrow} \mathcal{N} \models \psi(a, b)$
$\Rightarrow \mathcal{N} \models \varphi(a)$

(2) $\Rightarrow$ (1): We show by induction on the complexity of $\alpha(z)$, $z = (z_1, \ldots, z_n)$, and $c \in M^n$,

(*) $\mathcal{M} \models \alpha(c) \Leftrightarrow \mathcal{N} \models \alpha(c)$.

- α is atomic. Then α is quantifier free, and (*) is always true (whenever $\mathcal{M} \subseteq \mathcal{N}$)
- $\neg, \vee, \wedge$ are easy
- $\alpha(z)$ is $\exists x \beta(z, x)$. Let $\varphi(x)$ be the L_M-formula $\beta(\underline{c}, x)$.

$$\begin{aligned} \mathcal{N} \models \alpha(c) &\Leftrightarrow \mathcal{N} \models \exists x \varphi(x) \\ &\overset{(2)}{\Leftrightarrow} \mathcal{N} \models \varphi(a) \text{ for some } a \in M \\ &\Leftrightarrow \mathcal{N} \models \beta(c, a) \\ &\Leftrightarrow \mathcal{M} \models \beta(c, a) \text{ by induction} \\ &\Leftrightarrow \mathcal{M} \models \alpha(c). \end{aligned}$$

2015 02 25

Can rephrase the definition of $\preccurlyeq$ as follows:

$\mathcal{M} \preccurlyeq \mathcal{N}$ (if and only) if they module the same L_M-sentences.

We say $\mathcal{M}$ models σ if $\mathcal{M} \models \sigma$.

Point: Every L_M-sentence σ is of the form $\varphi(a_1, \dots, a_n)$ where $\varphi(x_1, \dots, x_n)$ is an L-sentence and $a_1, \dots, a_n \in M$. So $\mathcal{M} \models \sigma$ if and only if $\mathcal{M} \models \varphi(a_1, \dots, a_n)$.

Proposition: Isomorphism are elementary. (Much wow.)

Proof: $j: \mathcal{M} \to \mathcal{N}$ an L-isomorphism, i.e., j is a surjective L-embedding. Prove by induction on complexity of $\varphi(x)$, $x = (x_1, \dots, x_n)$ and all $a \in M^n$.

(*) $\mathcal{M} \models \varphi(a)$ if and only if $\mathcal{N} \models \varphi(ja)$

- φ is atomic: prop 4.22 since φ is quantifier free
- $\vee, \neg, \wedge$ ✓
- $\varphi(x)$ is $\exists y \psi(x, y)$

$$\begin{aligned}
\mathcal{M} \models \varphi(a) &\iff \text{there is } b \in M \text{ st } \mathcal{M} \models \psi(a, b) \\
\text{ind.} \to &\iff \text{there is } b \in M \text{ st } \mathcal{N} \models \psi(ja, jb) \\
\text{surj.} \to &\iff c \in N \text{ st } \mathcal{N} \models \psi(ja, c) \\
&\iff \mathcal{N} \models \exists y \psi(ja, y) \\
&\iff \mathcal{N} \models \varphi(ja)
\end{aligned}$$

∎

Def $\mathcal{M}$ L-structure, $B \subseteq M$

$\mathrm{Aut}_B(\mathcal{M}) = \{$set of automorphisms $j: \mathcal{M} \to \mathcal{M}\}$ that fix B pointwise $\}$

B-automorphisms of $\mathcal{M}$. (automorphism = isomorphism to self)

Corollary: $X \subseteq M^n$ is B-definable, $j \in \mathrm{Aut}_B(\mathcal{M})$

Then $j(X) = X$.

Proof: There is an L-formula $\varphi(x_1, \dots, x, y_1, \dots, y)$

This can be used sometimes to show that certain sets are not definable.

Example: Consider $\mathcal{R} = (\mathbb{R}, <)$.

(a) $(0,1)$ is not $\emptyset$-definable

$j : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$ is an L-automorphism and it doesn't fix $(0,1)$ setwise.

(b) $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not definable in $\mathcal{R}$.

(Recall: a function $f : X \to Y$, where $X \subseteq M^n$, $Y \subseteq M^\ell$ is B-definable if its graph $\Gamma(f) \subseteq X \times Y \subseteq M^{n+\ell}$ is B-definable.)

Suppose $+$ is definable. Then it's B-definable for some finite $B \subseteq \mathbb{R}$, $B = \{b_1, \ldots, b_m\}$, $b_1 < \cdots < b_m$.

Choose $c > \max\{b_m, 0\}$. Define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} x & \text{if } x \le c, \\ \frac{x+c}{2} & \text{if } x > c. \end{cases}$$

Then j is an L-automorphism. It fixes B pointwise but

$$j(\underbrace{c+1, c+1, 2c+2}_{\in \Gamma(+)}) = (\underbrace{c+\tfrac{1}{2}, c+\tfrac{1}{2}, \tfrac{3}{2}c+1}_{\notin \Gamma(+)})$$

So $\Gamma(+)$ is <u>not</u> B-definable.

Exercise: $\mathbb{R}$ is not definable in $(\mathbb{C}, 0, 1, +, -, \times)$.

<u>Downward Lowenheim-Skolem Theorem:</u>

$\mathcal{N}$ L-structure, $A \subseteq N$, then there exists an elementary substructure $\mathcal{M} \preceq \mathcal{N}$ with $A \subseteq M$ and

$$|M| \le |A| + |L| + \aleph_0.$$

Remarks:

- the cardinal sum is just max
- sharp:
  - can't expect $|M| < |A|$ since $A \subseteq M$
  - if $|M| < \aleph_0$ then let σ say "I have size $|M|$" so $\mathcal{M} = \mathcal{N}$
  - can't expect $|M| < |L|$
- when L is countable, DLS implies every struct. has a countable elementary substructure ($A = \emptyset$

"Skolemheim"

Proof: $\kappa = |A| + |L| + \aleph_0$. Build recursively a countable chain of subsets of N, $A = A_0 \subseteq A_1 \subseteq \cdots$ such that

- $|A_n| \le \kappa \quad \forall n \ge 0$
- for all $n \ge 0$, and any L_{A_n}-formula $\varphi(x)$ in one variable x, if $\mathcal{N} \models \exists x \varphi(x)$ then this is witnessed by an element in A_{n+1}

Construct A_{n+1} by adding to A_n a realization for each L_{A_n}-formula $\varphi(x)$ that has a solution (in $\mathcal{N}$).

$|A_{n+1}| \le |\text{set of } L_{A_n}\text{-formulas}| \le |\text{finite strings from } L_{A_n} \text{ plus ctbly many other symbols,}$ Var, (, ... etc

$$= \sum_{\ell < \aleph_0} \kappa^\ell = \kappa \qquad \text{as } \kappa \ge \aleph_0$$

Let

$$M = \bigcup_{n < \aleph_0} A_n.$$

If $\varphi(x)$ is an L_M-formula in one variable with a realization in $\mathcal{N}$, then it has a realization in M (it is an L_{A_m}-formula for some m). By A3Q7, M is the universe of an elementary substructure of $\mathcal{N}$. ◻

<u>Def</u>] L language. An <u>L-theory</u> is a set of L-sentences. A <u>model</u> of an L-theory is an L-structure in which these sentences are true. An L-theory is <u>consistent</u> if it has a model.

A class K of L-structures is said to be <u>elementary</u> or <u>axiomatizable</u> if there is an L-theory T such that

$$K = \text{class of all models of } T =: \mathrm{Mod}(T).$$

We write $\mathcal{M} \models T$ to mean $\mathcal{M} \models \sigma$ for all $\sigma \in T$.

Examples:

(a) $L = \{e, \cdot, \mathrm{inv}\}$

The axiom of groups form an L-theory

- abelian groups are elementary

"perhaps an improvement of remark 7 is remark d"

he meant c not 7

elementary: exponent

- torsion-free groups
- groups of ~~order~~ fixed n
- divisible groups
- infinite groups

not elementary:

- torsion groups
- finite groups (prove later)

2015 03 02

<u>Def</u>] L-structure $\mathcal{M}$.

$Th(\mathcal{M})$ is the set of all L-sentences true in $\mathcal{M}$.

If $\mathcal{N}$ is another L-structure, then $\mathcal{M}$ is <u>elementarily equivalent</u> to $\mathcal{N}$, $\mathcal{M} \equiv \mathcal{N}$ if $Th(\mathcal{M}) = Th(\mathcal{N})$. ie, for all L-sentences σ,

$$\mathcal{M} \models \sigma \iff \mathcal{N} \models \sigma.$$

Remarks:

(a) If $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$. This is because by 4.24 L-isomorphisms are elementary L-embeddings, and apply $n=0$ of definition of elementary L-embedding.

(b) If M is finite and $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{M} \cong \mathcal{N}$.

In general (infinite case) $\equiv$ is a much coarser notion than $\cong$.

(c) Suppose $\mathcal{M} \subseteq \mathcal{N}$. Then

$$\mathcal{M} \preceq \mathcal{N} \iff \mathcal{M}_M \equiv \mathcal{N}_M \text{ as } L_M\text{-structures}$$

(restatement of definition of $\preceq$).

<u>Example</u>: $\subseteq + \equiv \not\Rightarrow \preceq$

$\mathcal{M} = (\mathbb{N} \setminus \{0\}, <)$

$\mathcal{N} = (\mathbb{N}, <)$

$\mathcal{M} \subseteq \mathcal{N}$ $\quad$ $\mathcal{M} \equiv \mathcal{N}$ since $\mathcal{M} \cong \mathcal{N}$ ($x \mapsto x-1$ is an L-iso)

$\mathcal{M} \not\preceq \mathcal{N}$ as $\exists y\,(y<1)$ L_M-sentence

(d) $\mathcal{M}, \mathcal{N}$ L-structures. There exists an elementary embedding $j: \mathcal{M} \to \mathcal{N} \iff \mathcal{N}$ can be extended to a model of $Th(\mathcal{M}_M)$ in L_M

Proof: Leave details to you. If $j: \mathcal{M} \to \mathcal{N}$ is elementary then define whisper → "Jesus" breaks chalk twice in a row

"you have to say 1 plus 1 is not 2"

2015 03 02(2

the L_M-structure $\mathcal{N}' = (N, \ldots)$ where L is interpreted as it is in $\mathcal{N}$

for each $a \in M$,

$a^{\mathcal{N}'} := j(a)$

This works: $\mathcal{N}' \models Th(\mathcal{M}_M)$

Conversely, suppose $\mathcal{N}'$ is an expansion of $\mathcal{N}$ to an L_M-structure such that $\mathcal{N}' \models Th(\mathcal{M}_M)$. Define $j: \mathcal{M} \to \mathcal{N}$ by $j(a) = a^{\mathcal{N}'}$. This works: j is elementary $\square$

<u>Def</u>) Given an L-theory T, and an L-sentence σ. Then we say that T <u>implies</u> σ (or T <u>entails</u> σ or σ is a <u>consequence</u> of T) if $\mathcal{M} \models \sigma$ for every model $\mathcal{M} \models T$. Notation: $T \models \sigma$.

A theory is <u>complete</u> if for every L-sentence, σ, either $T \models \sigma$ or $T \models \neg\sigma$.

Example

(a) $\mathcal{M}$ L-structure $Th(\mathcal{M})$ is complete

(b) Let T_1 be the theory of rings in the language $L = \{0, 1, +, -, \times\}$. Then T_1 is incomplete ($\forall x \exists y (y^2 = x)$ or $\forall x (x \neq 0 \Rightarrow \exists y (xy = 1))$)

Let ACF be the theory of algebraically closed fields (infinite list of axioms, in particular, one for each polynomial)

$1+1=0$ true in $\mathbb{F}_2^{alg}$, false in $\mathbb{Q}^{alg}$, so ACF is incomplete

Fact (prove later) ACF_0, theory of algebraically closed fields of characteristic 0, and ACF_p, " of " p, are complete theories.

Lemma: Suppose T is ~~an~~ consistent L-theory. Let T' = set of all consequences of T. The following are equivalent:

(1) T is complete

(2) T' is maximally consistent

(3) $T' = Th(\mathcal{M})$ for some, equivalently, for all, $\mathcal{M} \models T$.

(4) Any two models of T are elementarily equivalent.

Proof:

(i)$\Rightarrow$(ii): T' is consistent: T consistent so there is $\mathcal{M} \models T$, but then $\mathcal{M} \models T'$.

T' is maximally so: Let $T' \subsetneq S$ L-theory, let $\sigma \in S \setminus T'$
$\Rightarrow T \not\models \sigma \overset{(1)}{\Rightarrow} T \models \neg\sigma \Rightarrow \neg\sigma \in T' \subseteq S$
$\Rightarrow \sigma, \neg\sigma \in S$ so S has no model ie is inconsistent

(ii)$\Rightarrow$(iii): Suppose $\mathcal{M} \models T$. Then $\mathcal{M} \models T' \Rightarrow T' \subseteq Th(\mathcal{M})$
$\overset{(2)}{\Rightarrow} T' = Th(\mathcal{M})$

(iii)$\Rightarrow$(iv) Suppose $T' = Th(\mathcal{M})$ for some $\mathcal{M} \models T$. It suffices to show that for all $\mathcal{N} \models T$, $\mathcal{N} \equiv \mathcal{M}$. Let $\mathcal{N} \models T \Rightarrow$
$\mathcal{N} \models T' \Rightarrow T' \subseteq Th(\mathcal{N}) \Rightarrow Th(\mathcal{M}) \subseteq Th(\mathcal{N})$
$Th(\mathcal{M})$ is complete so (i)$\Rightarrow$(ii) gives $\mathcal{M}$ max. cons. so
$Th(\mathcal{M}) = Th(\mathcal{N})$. $\therefore \mathcal{M} \equiv \mathcal{N}$

(iv) Suppose $T \not\models \sigma$. So there is $\mathcal{M} \models T$, $\mathcal{M} \models \neg\sigma$.
Let $\mathcal{N}$ be any model of T. By (4), $\mathcal{N} \equiv \mathcal{M}$, so
$\mathcal{N} \models \neg\sigma$. $\therefore T \models \neg\sigma$. T is complete. $\square$

Theorem: (Compactness of first order logic)
L language, T an L-theory. T is consistent if and only if every finite subset of T is consistent.

Remark: Immediate consequence of "completeness" (see PMATH 432).
But we give an "algebraic" proof using ultraproducts.

Motivating the proof:
($\Leftarrow$) For every finite $\Sigma \subseteq T$ we have a model $\mathcal{M}_\Sigma \models \Sigma$. We want to build a model of T.

$$\{\mathcal{M}_\Sigma, \Sigma \subseteq T \text{ finite}\}.$$

Idea: Treat the structures as better and better "approximations" to a model of T.
Suppose $\Sigma \subseteq T$ finite, if $\mathcal{M}_\Sigma \models T$ done. If $\mathcal{M}_\Sigma \not\models T$ then $\sigma \in T$ such that $\mathcal{M}_\Sigma \models \neg\sigma$. But then $\Sigma \cup \{\sigma\} \subseteq T$ finite, we should consider $\mathcal{M}_{\Sigma\cup\{\sigma\}}$ instead.
Want to take a "directed limit" of the lattice of L-structures

$$\{\mathcal{M}_\Sigma, \Sigma \subseteq T \text{ finite}\}$$

2015 03 02

Ultra Products

Def I non-empty set. $\mathcal{F} \subseteq \mathbb{P}(I)$ is a <u>filter on I</u> if
(i) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$;
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;
(iii) if $A \in \mathcal{F}$ and $A \subseteq B \in \mathbb{P}(I)$ then $B \in \mathcal{F}$.

Examples:
(a) $I = \mathbb{R}$, $\mathcal{F} = \{A \subseteq \mathbb{R};\ \mathbb{R} \setminus A$ has Lebesgue measure $0\}$
(b) I any infinite set, $\aleph_0 \leq \kappa \leq |I|$.
$$\mathcal{F} = \{A \subseteq I;\ |I \setminus A| < \kappa\}.$$
In particular, take $I = \omega$, $\kappa = \aleph_0$.
$$\mathcal{F} = \{A \subseteq \omega;\ A \text{ is cofinite}\}.$$
This is called Frechét filter on ω.
(c) <u>Principal filter</u>: I any non-empty set, fix $x \in I$.
$$\mathcal{F} := \{A \subseteq I;\ x \in A\}.$$

2015 03 04

An <u>ultrafilter</u> is a maximal filter.

Lemma: I infinite set, $\mathcal{U}$ a filter on I.
$\mathcal{U}$ is an ultrafilter if and only if for every $A \subseteq I$ either A or $I \setminus A$ is in $\mathcal{U}$.

Proof: ($\Leftarrow$): $\mathcal{U} \subsetneq \mathcal{F}$. let $A \in \mathcal{F} \setminus \mathcal{U}$.
if $I \setminus A \in \mathcal{U} \Rightarrow I \setminus A \in \mathcal{F}$
$\Rightarrow \emptyset = A \cap (I \setminus A) \in \mathcal{F}$ ※ So $A \notin \mathcal{U}$ and $I \setminus A \notin \mathcal{U}$.
($\Rightarrow$): $\mathcal{U}$ ultrafilter, $A \notin \mathcal{U}$. Want $I \setminus A \in \mathcal{U}$.
$$\mathcal{F} = \{X \subseteq I,\ X \supseteq Y \setminus A \text{ for some } Y \in \mathcal{U}\}$$
Check: $\mathcal{U} \subseteq \mathcal{F}$, $\mathcal{F}$ is a filter. By maximality, $\mathcal{U} = \mathcal{F}$.
$\therefore I \setminus A \in \mathcal{U}$. ∎

I infinite set, $\mathcal{F}$ a filter on I. Consider the poset
$$\{\mathcal{G} \subseteq \mathbb{P}(I);\ \mathcal{G} \text{ is a filter on } I \text{ and } \mathcal{G} \supseteq \mathcal{F}\}.$$
under $\subseteq$. Check the union of a chain of filters containing $\mathcal{F}$ is a filter containing $\mathcal{F}$.

So Zorn's lemma gives us a maximal filter containing $\mathcal{F}$.

Proposition: Every filter can be extended to an ultra filter.

Remark: Principal filters are maximal.

I infinite set, $\mathcal{U}$ ultrafilter on I, Sequence of L-structures $(\mathcal{M}_i ; i \in I)$.
We define the <u>ultraproduct of $(\mathcal{M}_i ; i \in I)$</u> to be the following L-structure,

$$\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$$

universe:

$$\bigtimes_{i \in I} M_i / E$$

where E is the equivalence relation $f E f'$ if

$$\{ i \in I ; f(i) = f'(i) \} \in \mathcal{U}.$$

(Here we view elements of $\times_{i\in I} M_i$ as functions on I, f, with $f(i) \in M_i$. We sometimes view f as the sequence $(f(i); i \in I)$.)
Since $\mathcal{U}$ is a filter, E is an equivalence relation. Denote elements of M by $[f]$.

Interpretation in $\mathcal{M}$:

$c \in L^{con}$, $\quad c^{\mathcal{M}} := [(c^{\mathcal{M}_i} ; i \in I)]$

$f \in L^{fun}$ n-ary, $[a_1], \ldots, [a_n]$, $a_k \in \times_{i\in I} M_i$

$$f^{\mathcal{M}}([a_1], \ldots, [a_n]) = [(f^{\mathcal{M}_i}(a_1(i), \ldots, a_n(i)) ; i \in I)]$$

$R \in L^{rel}$ n-ary, $[a_1], \ldots, [a_n] \in M$

$$([a_1], \ldots, [a_n]) \in R^{\mathcal{M}} \iff \{ i ; (a_1(i), \ldots, a_n(i)) \in R^{\mathcal{M}_i} \} \in \mathcal{U}.$$

Need to check that these definitions are independent of the choice of a representative.

Proof: Suppose $a_1 E a_1', \ldots, a_n E a_n'$. Let, for $k = 1, \ldots, n$,

$$A_k := \{ i \in I ; a_k'(i) = a_k(i) \} \in \mathcal{U}.$$

So $A_1 \cap \cdots \cap A_n \in \mathcal{U}$.

$$B := \{ i ; (a_1(i), \ldots, a_n(i)) \in R^{\mathcal{M}_i} \} \in \mathcal{U}$$

?

$\iff B \cap A_1 \cap \cdots \cap A_n \in \mathcal{U}$ $\qquad$ let $B' = \{ i \in I ; (a_1'(i), \ldots, a_n'(i)) \in R^{\mathcal{M}_i} \} \in \mathcal{U}$

$\iff B' \cap A_1 \cap \cdots \cap A_n \in \mathcal{U} \iff B' \in \mathcal{U}$

2015 03 04

Remark: For this definition, $\mathcal{U}$ need only be a filter. We haven't used maximality.

Ex. If $\mathcal{U}$ is a principal ultrafilter then

$$\prod_{\mathcal{U}} \mathcal{M}_i$$

is isomorphic to some $\mathcal{M}_j$, $j \in I$.

Theorem [Łoś' Theorem]:
Suppose $\mathcal{U}$ is an ultrafilter on I, $(\mathcal{M}_i ; i \in I)$ sequence of L-structures,

$$\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i,$$

$\phi(x_1, \dots, x_n)$ L-structure, $[a_1], \dots, [a_n] \in M$,

$$\mathcal{M} \models \phi([a_1], \dots [a_n]) \iff \{i \in I ; \mathcal{M}_i \models \phi(a_1(i), \dots, a_n(i))\} \in \mathcal{U}.$$

In particular, when $n = 0$, $\mathcal{M} \models \sigma$ if and only if $\{i \in I ; \mathcal{M}_i \models \sigma\} \in \mathcal{U}$.

Proof: By induction on complexity of ϕ.
Claim: $t(x_1, \dots x_n)$ is an L-term then

$$t^{\mathcal{M}}([a_1], \dots, [a_n]) = [(t^{\mathcal{M}_i}(a_1(i), \dots, a_n(i)) ; i \in I)].$$

For constants and basic functions, this is the definition of $\mathcal{M}$.
Use induction to prove it for all t. (Exercise)

- ϕ is atomic
  - ϕ is $t(x_1, \dots, x_n) = s(x_1, \dots, x_n)$, s, t L-terms

$$\begin{aligned}
\mathcal{M} \models \phi([a_1], \dots [a_n]) &\iff t^{\mathcal{M}}([a_1], \dots, [a_n]) = s^{\mathcal{M}}([a_1], \dots, [a_n]) \\
&\iff [(t^{\mathcal{M}_i}(a_1(i), \dots, a_n(i)) ; i \in I)] = [(s^{\mathcal{M}_i} \dots)] \\
&\iff \{i \in I ; t^{\mathcal{M}_i}(a_1(i), \dots) = s^{\mathcal{M}_i}(a_1(i), \dots)\} \in \mathcal{U}
\end{aligned}$$

$$\quad = \{i \in I ; \mathcal{M}_i \models \phi(a_1(i), \dots, a_n(i))\}$$

  - check other atomic case
- if ϕ is $\neg\psi$:

$$\begin{aligned}
\mathcal{M} \models \phi([a_1] \dots [a_n]) &\iff \mathcal{M} \not\models \psi([a_1], \dots) \\
&\iff \{i ; \mathcal{M}_i \models \psi(a_1(i), \dots)\} \notin \mathcal{U} \\
\text{as ultra} \longrightarrow &\iff \{ \quad \models \phi(\quad \} \in \mathcal{U}
\end{aligned}$$

· check: $\wedge$, $\vee$

· if $\phi(x_1,\ldots,x_n)$ is $\exists y\, \psi(x_1,\ldots,x_n,y)$

$\mathcal{M} \models \phi([a_1],\ldots,[a_n]) \iff$ there is $[b] \in M$ st $\mathcal{M} \models \psi([a_1],\ldots,[a_n],[b])$

$\iff$ " $X_b \in \mathcal{U}$

$X_b = \{i \in I;\ \mathcal{M}_i \models \psi(a_1(i),\ldots,a_n(i),b(i))\}$

$Y = \{i \in I;\ \mathcal{M}_i \models \phi(a_1(i),\ldots,a_n(i))\}$

$= \{i \in I;$ there is $c \in M_i$ st $\mathcal{M} \models \psi(a_1(i),\ldots,a_n(i),c)\}$

for any $b \in M$, $X_b \underset{\text{sct}}{\subseteq} Y_{c=b(i)}$ so we get $(\Rightarrow)$ (as $X_b \in \mathcal{U}$ and $X_b \subseteq Y \Rightarrow Y \in \mathcal{U}$)

For the converse, assume $Y \in \mathcal{U}$, and seek a b st $X_b \in \mathcal{U}$. For each $i \in Y$, let $b_i \in M_i$ be such that $\mathcal{M}_i \models \psi(a_1(i),\ldots,a_n(i),b_i)$. For $i \notin Y$, let b_i be arbitrary. Let

$$b \in \bigtimes_{i \in I} M_i$$

be st $b(i) = b_i$. Note $Y \subseteq X_b$. So $X_b \in \mathcal{U}$. ∎

A special case is when each $\mathcal{M}_i = \mathcal{N}$. Then we denote

$$\prod_{\mathcal{U}} \mathcal{N} = \mathcal{N}^I/\mathcal{U}$$

and call it the <u>ultrapower</u> of $\mathcal{N}$ with respect to $\mathcal{U}$. Consider

$$\begin{aligned} d: \mathcal{N} &\longrightarrow \mathcal{N}^I/\mathcal{U} \\ a &\longmapsto [f(a)] \quad \text{where } f(i) = a \end{aligned}$$

This map is an elementary embedding, by Łos' theorem. (check) So identifying $\mathcal{N}$ with its image we obtain a way of producing elementary extensions of arbitrary structures.

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Proposition: Suppose

$$\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$$

on ω

is a non-principal ultraproduct. Then $\mathcal{M}$ is <u>$\aleph_1$-compact</u>: If $\{F_i ; i<\omega\}$ are definable subsets of M^n with <u>finite intersection property</u> ($\bigcap_{i=0}^m F_i \neq \emptyset$ for all $m>0$) then $\bigcap_{i<\omega} F_i \neq \emptyset$.

Note: $(\mathbb{Z},<)$ is not $\aleph_1$-compact, neither is $(\mathbb{R},<)$, as
$$\bigcap_{m\geq 0}(m,\infty)=\emptyset$$

Proof: For convenience suppose $n=1$. Each $F_i\subseteq M$ is defined, $\varphi_i(x)$ L formula. We may assume $\models \varphi_{i+1}(x)\longrightarrow\varphi_i(x)$, $\varphi_0(x)$ is $x=x$. For each i, let

– cheating; should be L_M

$$n_i=\max\{n\leq i\ ;\ \mathcal{M}_i\models\exists x\varphi_n(x)\}.$$
So n_i exists, $1\leq n_i\leq i$. We are trying to find $a\in M$ such that
$$a\in\bigcap_{n???}F_{n_j}$$
ie such that
$$\mathcal{M}\models\varphi_n(a)$$
for all $n<\omega$.

Define a sequence $(a_i)_{i<\omega}$ by choosing $a_i\in M_i$ st $\mathcal{M}_i\models\varphi_{n_i}(a_i)$.

Let $a=[(a_i)_{i<\omega}]\in M$. Fix $n<\omega$.
$$X_n:=\{i\ ;\ i\geq n\ \wedge\ \mathcal{M}_i\models\exists x\varphi_n(x)\}$$
Claim: $X_n\in\mathcal{U}$.

Verification: $F_n\neq\emptyset$. (FIP) $\Rightarrow\mathcal{M}\models\exists x\varphi_n(x)$.

$\overset{Łoś}{\Longrightarrow}\{i\in\omega\ ;\ \mathcal{M}_i\models\exists x\varphi_n(x)\}\in\mathcal{U}$.

Also $\{i\in\omega\ ;\ i\geq n\}\in\mathcal{U}$ by non-principality. $\therefore X_n\in\mathcal{U}$. □

Claim: $X_n\subseteq\{i<\omega\ ;\ \mathcal{M}_i\models\varphi_n(a_i)\}$

Verification: $i\in X_n\Rightarrow\mathcal{M}_i\models\exists x\varphi_n(x)$ and $i\geq n$

$\Rightarrow n_i\geq n$ by max choice of n_i

$\Rightarrow\varphi_{n_i}(x)\longrightarrow\varphi_n(x)$

By choice of a_i, $\mathcal{M}_i\models\varphi_{n_i}(a_i)\Rightarrow\mathcal{M}_i\models\varphi_n(a_i)$. □

So $\{i<\omega\ ;\ \mathcal{M}_i\models\varphi_n(a_i)\}\in\mathcal{U}$

$\overset{Łoś}{\Longrightarrow}\mathcal{M}\models\varphi_n(a)$.

$\Rightarrow a\in F_n$.

But n was arbitrary. ⊠

Example: "Nonstandard analysis"

$\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \times, <)$

$\mathcal{U}$ non-principal ultrafilter on ω.

Consider

$$\mathcal{R}^* = \mathcal{R}^\omega / \mathcal{U} = \prod_{\mathcal{U}} \mathcal{R}$$

So $\mathcal{R}^* = (\mathbb{R}^*, 0, 1, +, -, \times, <)$ is ordered field extension of the reals satisfying all the same $L_{\mathbb{R}}$-sentences. But ~~$\mathbb{R}^*$~~ $\mathcal{R}^*$ is $\aleph_1$-compact. In particular, $\mathbb{R}^*$ has <u>infinite</u> elements $\alpha > \mathbb{Z}$. $\mathbb{R}^*$ has <u>infinitesimal</u> elements, for any $r \in \mathbb{R}$ there is $\alpha \in \mathbb{R}^*$ with $\alpha \in (r, r + \frac{1}{n})$ for all $n > 0$.

Compactness Theorem:

T an L-theory. If every finite subset of T is consistent then T is consistent.

Proof:
$$I = \mathcal{P}^{fin}(T) = \{\Sigma \subseteq T ; \Sigma \text{ finite}\}.$$

If $\Sigma \in I$, $X_\Sigma := \{\Sigma' \in I ; \Sigma \subseteq \Sigma'\}$.

$$\mathcal{A} = \{X_\Sigma ; \Sigma \in I\} \subseteq \mathcal{P}(I).$$

(i) $\emptyset \in \mathcal{A}$, $I = X_\emptyset \in \mathcal{A}$

(ii) $X_\Sigma \cap X_\Delta = X_{(\Sigma \cup \Delta)} \in \mathcal{A}$.

Check $\mathcal{F} = \{Y \subseteq I ; Y \supseteq X_\Sigma \text{ some } \Sigma \in I\}$ is a filter. Extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on I.

(Remark: If T is infinite then $\mathcal{U}$ is non-principal, since if $\Sigma \in I$ then $\Sigma \notin X_{\{\sigma\}} \in \mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ for $\sigma \in T \setminus \Sigma$ (possible as T inf., Σ finite).)

By assumption, for all $\Sigma \in I$, there is a model $\mathcal{M}_\Sigma \models \Sigma$.

Let
$$\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_\Sigma$$

We show ~~so~~ $\mathcal{M} \models T$. ~~...~~ Let $\sigma \in T$.

$$X_\sigma \subseteq \{\Sigma \in I ; \mathcal{M}_\Sigma \models \sigma\}$$

As $X_\sigma \in \mathcal{U}$, $\{\Sigma \in I ; \mathcal{M}_\Sigma \models \sigma\} \in \mathcal{U}$.

$\overset{Łoś}{\Longrightarrow} \mathcal{M} \models \sigma \qquad \therefore \mathcal{M} \models T$

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Corollary: $T \models \sigma$ if and only if for some finite $\Sigma \subseteq T$, $\Sigma \models \sigma$.

Proof: Note in general $S \models \tau$ if and only if $S \cup \{\neg\tau\}$ is inconsistent. ∎

Examples: $L = \emptyset$. The class of all infinite L-structures, is not finitely axiomatizable

Proof: This class is axiomatizable: for each $n > 1$, let

$$\tau_n = \exists x_1 \cdots x_n \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \right)$$

τ_n "says" the universe has at least n elements. So

$$T = \{\tau_n ; n > 1\}$$

axiomatizes the class of infinite structures

Suppose $Mod(T)$ is finitely axiomatizable, say by single sentence σ. So $Mod(\{\sigma\}) = Mod(T)$. So $T \models \sigma$. By compactness,

$$\{\tau_1, \ldots, \tau_N\} \models \sigma \quad \text{for some } N \geq 1$$

Any finite set ~~with~~ of size $> N$ is a model of LHS but not ~~the~~ of RHS. Contradiction. ∎

$L = \{e, \cdot, inv\}$

Example: The class of torsion-free groups is not finitely axiomatizable.

Proof: $T = \{\tau_n ; n \geq 1\}$ + axioms for groups, τ_n: $\forall x (x \neq e \Rightarrow x^n \neq e)$.

Every finite subset of T has a model which is not torsion-free. ($\mathbb{Z}/p\mathbb{Z}$'s, p prime)

As above. ∎

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$L = \{e, \cdot, inv\}$

The class of torsion groups is not elementary.

Proof 1: Compactness

Suppose T axiomatizes the torsion groups. Let $L' = L \cup \{c\}$, a new constant symbol, $T' = T \cup \{\underbrace{c \cdots c}_{n} \neq e ; n > 0\}$. T' is inconsistent since if $\mathcal{M}' \models T'$ then $\mathcal{M}' = (e, \cdot, inv, c^{\mathcal{M}'} = a)$ $a \in M'$. The L-reduct of $\mathcal{M}'$, $\mathcal{M} = (M, e, \cdot, inv)$ is a model of T. $\#$

By compactness, some finite subset of T' has no model. In particular, there is an $N > 0$ st

$T \cup \{c^n \neq e ; n=1, \dots N\}$

is inconsistent. But $\mathcal{M} = (\mathbb{Z}/p\mathbb{Z}, p\mathbb{Z}, +, -, 1+p\mathbb{Z})$ is a model as long as $p > N$. $\times\!\!\!\times$ $\square$

Proof 2: ultraproducts

Suppose such a T exists. Let

$$\mathcal{M}_i := (\mathbb{Z}/i\mathbb{Z}, 0, +, -), \quad 0 < i < \omega$$

all models of T. Fix a non-principal ultrafilter $\mathcal{U}$ on ω.

Let

$$\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i .$$

By Łos, $\mathcal{M} \models T$, so $\mathcal{M}$ is Torsion. Let $a = [(1 \bmod i)_{i<\omega}^{\sim}]$.

For any n, $\{i ; n(1 \bmod i) \neq 0 \text{ in } \mathcal{M}_i\} \supseteq \{i ; i > n\} \leftarrow$ cofinite, in $\mathcal{U}$ by non-principality

So $na \neq 0$ in $\mathcal{M}$. So a is not torsion. $\times\!\!\!\times$

Exercise. The class of algebraically closed fields of characteristic 0 is not finitely axiomatizable.

The class of algebraic extensions of $\mathbb{Q}^{alg}$ is not elementary.

Proposition: $\mathcal{M} \equiv \mathcal{N} \iff$ ~~iff~~ there exists $\mathcal{R}$ and elementary embeddings

$$\mathcal{M} \xrightarrow{\preccurlyeq} \mathcal{R}$$
$$\mathcal{N} \xrightarrow{\preccurlyeq} \mathcal{R} .$$

Proof: $(\Leftarrow)$: ✓

$(\Rightarrow)$: Recall, $\exists \mathcal{M} \xrightarrow{\preccurlyeq} \mathcal{R}$ iff $\mathcal{R}$ can be expanded to a model of $Th(\mathcal{M}_M)$ ← 4.45

Let $T = Th(\mathcal{M}_M) \cup Th(\mathcal{N}_N)$ in $L_{M \cup N} := L \cup \{\underline{m} ; m \in M\} \cup \{\underline{n} ; n \in N\}$ ← all distinct

It suffices to prove that T is consistent.

(If $\mathcal{R}' \models T$, let $\mathcal{R}$ be the L-reduct of $\mathcal{R}'$.)

Let $\Sigma \subseteq T$ be finite. We may assume $\Sigma = \{\sigma, \tau\}$ where

$\sigma = \varphi(\underline{m}_1, \dots, \underline{m}_r) \in Th(\mathcal{M}_M)$, $\tau = \psi(\underline{n}_1, \dots, \underline{n}_s) \in Th(\mathcal{N}_N)$

where φ, ψ are L-formulas.

Certainly $\mathcal{M}_M \models \sigma$. $\mathcal{N}_N \models \tau \Rightarrow \mathcal{N}_N \models \psi(\underline{n}_1, \dots, \underline{n}_s)$

$\Rightarrow \mathcal{N} \models \exists x_1 \dots \exists x_s\, \psi(x_1, \dots, x_s)$

$\Rightarrow \mathcal{M} \vdash \exists x_1 \ldots \exists x_s \, \psi(x_1 \ldots x_s)$

Let $a_1, \ldots, a_s \in M$ be st $\mathcal{M} \models \psi(a_1, \ldots a_s)$. Expand $\mathcal{M}_M$ into an $L_{M \cup N}$-structure, say $\mathcal{S}$, by setting

$$\underline{n_i}^{\mathcal{S}} = a_i$$

and other $\underline{n}^{\mathcal{S}}$ arbitrarily. Then $\mathcal{S} \models \sigma$ and $\mathcal{S}' \models T$. ☒

Corollary (Upward L-S theorem)
If $\mathcal{M}$ is an infinite structure then for any $\kappa \geq |M| + |L|$, $\mathcal{M}$ has an elementary extension of size κ.

Proof: Let $L' = L_M \cup \{c_\alpha ; \alpha < \kappa\}$, new constant symbols. Let $T = Th(\mathcal{M}_M) \cup \{c_\alpha \neq c_\beta ; \alpha < \beta < \kappa\}$. Clearly a model $\mathcal{N}'$ of T will be an elementary extension of $\mathcal{M}$ with $|N| \geq \kappa$. 4.45
Claim: T is consistent.
Verification: $\Sigma \subseteq T$ is finite then $\Sigma \subseteq Th(\mathcal{M}_M) \cup \{c_{\alpha_i} \neq c_{\alpha_j}, \alpha_1 < \cdots < \alpha_N < \kappa, i \neq j\}$
Expand $\mathcal{M}_M$ into an L'-theory $\mathcal{S}$ by choosing $a_1, \ldots, a_N$ distinct elements of M, interpreting c_{α_i} by a_i, and whatever for rest of c_α's. Then $\mathcal{S} \models \Sigma$. By compactness, claim holds. □
Let $\mathcal{N}' \models T$. Let $\mathcal{N}$ be the L-reduct of $\mathcal{N}'$. So by 4.45, $\exists \mathcal{M} \xrightarrow{\preceq} \mathcal{N}$.
By choice of T, $|N| \geq \kappa$.
Let $A \subseteq N$ st $M \subseteq A$ and $|A| = \kappa$. Apply DLS to get $\mathcal{R} \preceq \mathcal{N}$, $R \supseteq A$, $|R| \leq |A| + |L| + \aleph_0 = \kappa$. But $A \subseteq R$ so $\kappa \leq |R|$. So $|R| = \kappa$.

$$\mathcal{M} \preceq \mathcal{N} \qquad \mathcal{R} \preceq \mathcal{N} \qquad M \subseteq R$$

This implies $\mathcal{M} \preceq \mathcal{R}$ (exercise). ☒

Corollary (Vaught's Test):
Suppose T is an L-theory all of whose models are infinite. Suppose for some infinite cardinal κ, all models of T of size κ are isomorphic. $\leftarrow$ T is κ-categorical
Then T is complete. $\kappa \geq |L|$.

Proof: $\mathcal{M}, \mathcal{N}$ two models. By ULS and DLS there exists an elementary extension/substructure of $\mathcal{M}$, say $\mathcal{M}'$, such that $|M'| = \kappa$.

Similarly $\mathcal{N}'$ is an elementary substructure/extension of size κ.

$$\mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}$$ ☒

Example $L = \emptyset$, $T =$ theory of infinite sets
T is $\aleph_0$-categorical (for any κ actually (infinite))

Example $L =$ language of rings. Let $T = ACF_p$

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Example: DLO is complete
$L = \{<\}$, DLO = theory of dense linear orderings without endpoints.

Proof: We show that DLO is $\aleph_0$-categorical. We give a "back and forth" construction. Let $(E,<)$ and $(F,<)$ be models of DLO that are both countable. Enumerate

$$E = \{a_i ; i < \omega\}$$
$$F = \{b_i ; i < \omega\}$$

Note: enumeration has nothing to do with the ordering.
We build a chain of order-preserving bijections

$$f_0 \subseteq f_1 \subseteq f_2 \subseteq \cdots$$

where $f_i : A_i \to B_i$, $A_i \subseteq E$ finite, $B_i \subseteq F$ finite, with

$$E = \bigcup_{i<\omega} A_i, \quad F = \bigcup_{i<\omega} B_i.$$

Once we have this, then

$$f := \bigcup_{i<\omega} f_i : E \to F$$

will be an L-isomorphism.
We build $f_i : A_i \to B_i$ recursively, with $f_0 = A_0 = B_0 = \emptyset$.
Odd step $n+1 = 2m+1$: We ensure $a_m \in A_{n+1}$.
If $a_m \in A_n$, then do nothing: $f_{n+1} = f_n$, $A_{n+1} = A_n$, $B_{n+1} = B_n$.
If not, set $A_{n+1} = A_n \cup \{a_m\}$. Three possibilities for the relation of a_m with A_n:
(i) $a_m < A_n$
(ii) $a_m > A_n$
(iii) there is $\alpha < \beta$ consecutive in A_n st $\alpha < a_m < \beta$

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If (i), choose $b \in F$ st $b < B_n$. } possible by no endpoints + B_n finite
If (ii), choose $b \in F$ st $b > B_n$ }
If (iii) then $f_n(\alpha) < f_n(\beta)$ and nothing in B_n is in between (as f_n is an iso). So choose $b \in F$ st $f_n(\alpha) < b < f_n(\beta)$ (possible by density).
Set $B_{n+1} = B_n \cup \{b\}$, $f_{n+1} = f_n \cup \{(a_m, b)\}$. Then f_{n+1} is order preserving
even step $n+1 = 2m+2$: We ensure $b_m \in B_{n+1}$. If $b_m \in B_n$, do nothing. Otherwise, choose $a \in E$ whose $<$-relation to A_n precisely that for b_m to B_n relative to f_n. Set $B_{n+1} = B_n \cup \{b\}$, $A_{n+1} = A_n \cup \{a\}$, $f_{n+1} = f_n \cup \{(a, b_m)\}$.

DLO has only infinite models.
Vaught $\Rightarrow$ DLO is complete. ∎

So $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$. $\Rightarrow$ completeness of $\mathbb{R}$ is not captured by $Th(\mathbb{R}, <)$.

What about $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$ (ie $Th(\mathbb{Q}, <)_{\mathbb{Q}} = Th(\mathbb{R}, <)_{\mathbb{Q}}$)?
Yes. Needs quantifier elimination.

<u>Example</u> $L = \{0, 1, +, -, \times\}$. Fix p a prime or zero.
ACF_p = theory of alg closed fields of char p

$\sigma_n \quad \forall y_0 \ldots \forall y_n \left(\left(\bigvee_{i=0}^{n} y_i \neq 0 \right) \to \exists x \, (y_0 + y_1 x + \cdots + y_n x^n = 0) \right)$.

We will show ACF_0 is complete.
Note: ACF_p is <u>not</u> $\aleph_0$-categorical.
$\rightarrow p=0 \quad \mathbb{Q}^{alg} \not\cong \mathbb{Q}(t)^{alg}$
We will show ACF_p is κ-categorical for any $\kappa > \aleph_0$.

$$\mathbb{F} = \begin{cases} \mathbb{Q} & \text{if } p = 0 \\ \mathbb{Z}/p\mathbb{Z} & \text{if } p \text{ prime} \end{cases} \qquad \text{prime field}$$

Fact: $K \models ACF_p$ uncountable then $tr.deg(K/\mathbb{F}) = |K|$
Sketch Proof. In general, if $B \subseteq K$ is finite then $|\mathbb{F}(B)^{alg}| = \aleph_0$ since there

are only countably many polynomials over $\mathbb{F}(B)$.
If $B \subseteq K$ is infinite $|\mathbb{F}(B)^{alg}| = |B|$, by counting poly's.
Let $B \subseteq K$ be a transcendence basis for $K/\mathbb{F}$. Then $K = \mathbb{F}(B)^{alg}$

K uncountable $\Rightarrow$ B is infinite

and $\operatorname{trdeg}(K/\mathbb{F}) = |B| = |\mathbb{F}(B)^{alg}| = |K|$

Let κ be uncountable, $K, L \models ACF_p$ of size κ. Then

$$\operatorname{trdeg}(K/\mathbb{F}) = \operatorname{trdeg}(L/\mathbb{F}) = \kappa.$$

(by fact) Let B be a tr. basis for $L/\mathbb{F}$ $\quad\quad \iota : B \to C$ bij.

" C " $K/\mathbb{F}$

$$L = \mathbb{F}(B)^{alg} \longrightarrow \mathbb{F}(C)^{alg} = K$$ by uniqueness of alg closures

$$\frac{P(b_1, \ldots, b_n)}{Q(b_1, \ldots, b_n)} \longmapsto \frac{P(\iota(b_1), \ldots)}{Q(\iota(b_1), \ldots)}$$

$\therefore ACF_p$ is κ-categorical $\forall \kappa > \aleph_0$. Also ACF_p has only infinite models.
Thus ACF_p is complete. ■

So $(\mathbb{Q}^{alg}, 0, 1, +, -, \times) \equiv (\mathbb{C}, 0, 1, +, -, \times)$.

$\preceq$? Yes (needs quantifier elimination)

Application to Algebraic Geometry.

Theorem (Lefschetz Principle) If σ is an L-sentence, $L = \{0, 1, +, -, \times\}$, then the following are equivalent

(i) $(K, 0, 1, +, -, \times) \models \sigma$ some $K \models ACF_0$ (or any)

(ii) $(\mathbb{C}, 0, 1, +, -, \times) \models \sigma$

(iii) $((\mathbb{Z}/p\mathbb{Z})^{alg}, 0, 1, +, -, \times) \models \sigma$ for all but finitely many primes p

(iv) " for infinitely many p.

Proof: (i) $\Leftrightarrow$ (ii) is ACF_0 is complete

(ii) $\Rightarrow$ (iii) ACF_0 complete $\underset{4436}{\Longrightarrow}$ ACF_0's consequences $= Th(\mathbb{C}, 0, 1, +, -, \times)$

$\Rightarrow ACF_0 \models \sigma \Rightarrow \exists$ finite $\Sigma \subseteq ACF_0$ st $\Sigma \models \sigma$.

$\Sigma \subseteq ACF \cup \{\tau_1, \ldots, \tau_N\}$ where τ_1 is $1 \neq 0$.

for all $p > N$, $((\mathbb{Z}/p\mathbb{Z})^{alg}, 0, 1, +, -, \times) \models \Sigma \models \sigma$.

(iii) $\Rightarrow$ (iv) ✓

"I'm not sure it's instructive, but it's not uninstructive"

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(iv) $\Rightarrow$ (ii): If $(\mathbb{C}, 0, 1, +, -, \times) \nvDash \sigma$ then

$\quad$ " $\models \neg\sigma$

$\overset{ii \Rightarrow iii}{\Longrightarrow} ((\mathbb{Z}/p\mathbb{Z})^{alg}, 0, +, -, \times) \models \neg\sigma \quad \forall p > N$ prime

So for only finitely many primes p,

$$((\mathbb{Z}/p\mathbb{Z})^{alg}, 0, 1, +, -, \times) \models \sigma. \quad \blacksquare$$

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<u>Def</u> An L-theory T has <u>quantifier elimination</u> (QE) if for every L-formula $\varphi(x_1, .., x_n)$, $n > 0$, there exists a quantifier-free (q.f.) formula $\psi(x_1, \dots, x_n)$ such that

$$T \models \forall x_1 \dots \forall x_n (\varphi(x_1, .., x_n) \leftrightarrow \psi(x_1, .., x_n)).$$

We say φ and ψ are <u>T-equivalent</u>.

Remarks:

(1) Why $n > 0$? If L has no constant symbols then there are no q.f. L-sentences. This is the only problem.

Ex: If T admits QE and L has a constant symbol, then every L-sentence is equivalent to a q.f. L-sentence.

(2) $\varphi(x_1, .., x_n)$ is T-equivalent to $\psi(x_1, .., x_n)$ if and only if in every model $\mathcal{M} \models T$, $\varphi^{\mathcal{M}} = \psi^{\mathcal{M}}$.

<u>Preliminaries on substructures</u>

<u>Def</u> $\mathcal{M}$ L-structure, $A \subseteq M$, the <u>substructure generated by A</u> is the smallest substructure containing A. We denote it by $\langle A \rangle$.

Remark: Let

$$\mathcal{F} = \{B \subseteq M ; A \subseteq B, B \text{ is the universe of a substructure of } \mathcal{M}\}$$

Then $\langle A \rangle = \bigcap \mathcal{F}$.

Lemma:

$$\langle A\rangle = \{ t^{\mathcal{M}}(a_1,\dots,a_n);\ \text{all } n\geq 0,\ \text{all } L\text{-terms } t(x_1,\dots,x_n) \text{ and all } a_i\in A\}.$$

Proof: Exercise

Lemma: Suppose $\mathcal{M},\mathcal{N}$ are L-structures, $A\subseteq M$. Assume either $A\neq\emptyset$ or L has a constant symbol. Then the following are equivalent

(i) There is an L-embedding $\langle A\rangle \xrightarrow{j} \mathcal{N}$;

(ii) $\mathcal{N}$ can be expanded to a model of

Sometimes denoted $\mathrm{Diag}(\langle A\rangle)$ $\longrightarrow$ $q.f.\,Th(\mathcal{M}_A) = \{q.f.\ L_A\text{-sentences true in } \mathcal{M}_A\}$.

Remark: (1) If $\mathcal{B}$ is any substructure with $A\subseteq\mathcal{B}$, then

$$q.f.\,Th(\mathcal{M}_A) = q.f.\,Th(\mathcal{B}_A).$$

In particular, we can replace $\mathcal{M}$ by $\langle A\rangle$ in (ii).

(2) Compare to 4.45.

(3) Stated slightly differently in the notes (6.5)

Proof:

(i)$\Rightarrow$(ii) Given $j:\langle A\rangle\to\mathcal{N}$. Define $\mathcal{N}'$ by $\underline{a}^{\mathcal{N}'} = j(a)$ for all $a\in A$. Check $\mathcal{N}'\models q.f.\,Th(\mathcal{M}_A)$.

(ii)$\Rightarrow$(i): $\mathcal{N}'$ expands $\mathcal{N}$ and $\mathcal{N}'\models q.f.\,Th(\mathcal{M}_A)$. Define $j:\langle A\rangle\to\mathcal{N}$: If $c\in\langle A\rangle$ then by the previous lemma $c = t^{\mathcal{M}}(a_1,\dots,a_n)$ for some L-term t, $a_1,\dots,a_n\in A$:

$$j(c) := t^{\mathcal{N}}(\underline{a}_1^{\mathcal{N}'},\dots,\underline{a}_n^{\mathcal{N}'}).$$

Check j is an L-embedding. $\blacksquare$

Theorem. T an L-theory, $\varphi(x)$ an L-formula, $x=(x_1,\dots,x_n)$. Suppose either L has a constant symbol or $n>0$. Then the following are equivalent:

(i) $\varphi(x)$ is equivalent to a quantifier-free L-formula $\psi(x)$;

(ii) Given $\mathcal{M},\mathcal{N}\models T$ and a common substructure $\mathcal{B}\subseteq\mathcal{M}$, $\mathcal{B}\subseteq\mathcal{N}$.

(*) — Then for any $b\in\mathcal{B}^n$, $\mathcal{M}\models\varphi(b) \iff \mathcal{N}\models\varphi(b)$.

Proof:

(i)$\Rightarrow$(ii) Suppose $\varphi(x)$ is T-equivalent to $\psi(x)$. Then

$$\begin{aligned} \mathcal{M} \models \varphi(x) &\iff \mathcal{M} \models \psi(b) && \text{since } \mathcal{M} \models T \\ &\iff \mathcal{B} \models \psi(b) && \text{since } \psi \text{ is q.f. } (4.22(a)) \\ &\iff \mathcal{N} \models \psi(b) && '' \\ &\iff \mathcal{N} \models \varphi(b) && \text{since } \mathcal{N} \models T \end{aligned}$$

(ii)$\Rightarrow$(i) Consider

$$\Psi(x) = \{\psi(x);\ \psi(x) \text{ q.f. and } T \vdash \forall x(\varphi(x) \to \psi(x))\}.$$

We show that $\varphi(x)$ is T-consequence of $\Psi(x)$. Let $c = (c_1, \dots, c_n)$ be a tuple of new constant symbols, $L' = L \cup \{c_1, \dots, c_n\}$.

Claim: $T \cup \Psi(c) \models \varphi(c)$. (in L')

Verification: Suppose not. So there exists a model $\mathcal{M} \models T$ with $a = (a_1, \dots, a_n) \in M^n$ such that $\mathcal{M} \models \psi(a)$ for all $\psi \in \Psi$ but $\mathcal{M} \models \neg\varphi(a)$.

Let $\mathcal{B} = \langle a_1, \dots, a_n \rangle \subseteq \mathcal{M}$. We want to find an extension $\mathcal{N} \supseteq \mathcal{B}$ with $\mathcal{N} \models T$ such that $\mathcal{N} \models \varphi(a)$. This will contradict (ii), proving claim

Consider the $L_{\{a_1, \dots, a_n\}}$-theory

$$T' = T \cup \{\varphi(\underline{a})\} \cup \text{q.f.Th}(\mathcal{M}_{\{a_1, \dots, a_n\}}).$$

Subclaim. T' is consistent.

Subverification: Suppose not. By compactness,

$$T \cup \{\varphi(\underline{a})\} \cup \{\theta_1(\underline{a}), \dots, \theta_\ell(\underline{a})\}$$

is inconsistent for some $\ell \geq 0$, where $\theta_1(x), \dots, \theta_\ell(x)$ are q.f. L-formulas such that $\mathcal{M} \models \theta_i(a_1, \dots, a_n)$. So

$$T \models \varphi(\underline{a}) \to \bigvee_{i=1}^{\ell} \neg\theta_i(\underline{a})$$

Since $\underline{a}$ does not appear in L, we can interpret it any way we like. Thus

$$T \models \forall x(\varphi(x) \to \bigvee_{i=1}^{\ell} \neg\theta_i(x))$$

So $\bigvee_{i=1}^{\ell} \neg\theta_i(x) \in \Psi(x)$. Therefore $\mathcal{M} \models \bigvee_{i=1}^{\ell} \neg\theta_i(a)$. Contradiction. □

Let $\mathcal{N}$ be a model of T'. So by the previous lemma, there is an embedding $j: \langle a_1, \dots, a_n \rangle \to \mathcal{N}$ and $\mathcal{N} \models \varphi(j(a))$ and $\mathcal{N} \models T$.

Identifying $\mathcal{B}$ with its image under j we get

$$T \dashv \underset{\cup\!\!\!\!\!\!\;\;\mathcal{B}}{\mathcal{M}} \subseteq \mathcal{N} \models T \qquad \mathcal{M} \models \neg\varphi(a),\ \mathcal{N} \models \varphi(a)$$

This contradicts (ii). □

By compactness, $T \cup \Sigma(c) \models \varphi(c)$ where $\Sigma(x) \subseteq \Psi(x)$ is finite. Let

$$\sigma(x) = \bigwedge \Sigma(x).$$

"I should stop. Oh! Right, I didn't finish."

So $T \cup \{\sigma(c)\} \models \varphi(c)$, $T \models \sigma(c) \to \varphi(c)$. But c does not appear in L, so $T \models \forall x (\sigma(x) \to \varphi(x))$. But $T \models \forall x (\varphi(x) \to \sigma(x))$ since $\sigma(x) \in \Psi(x)$. Therefore $\varphi(x)$ is T-equivalent to $\sigma(x)$. ∎

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Corollary (Criterion for QE): T an L-theory satisfying:

(*) Whenever $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \le \mathcal{M}$, $\mathcal{A} \le \mathcal{N}$, $\psi(y)$ conjunction of L_A-literals (atomic or negated atomic L_A-formulas) in a single variable that has a realization in $\mathcal{M}$, then $\psi(y)$ has a realization in $\mathcal{N}$.

Then T admits QE.

Proof: Given $\varphi(x_1, \ldots, x_n)$, $n > 0$, we want to show by induction on the complexity of φ that φ is T-equivalent to a q.f. formula.

- φ is atomic ✓
- $\wedge, \vee, \neg$ ✓
- $\varphi(x)$, $x = (x_1, \ldots, x_n)$, is $\exists y\, \psi(x,y)$. By inductive assumption, $\psi(x,y)$ is T-equivalent to a quantifier-free formula $\psi'(x,y)$.

Exercise: $\psi'(x,y)$ is equivalent to a formula of the form

$$\bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell} \psi_{ij}(x,y), \quad \psi_{ij} \text{ } L\text{-literals} \qquad \text{(DNF)}$$

So $\varphi(x)$ is T-equivalent to $\underbrace{\exists y (\bigvee \bigwedge \psi_{ij}(x,y))}_{\varphi'(x)}$. We use thm 6.6 to prove that this is T-equivalent to a q.f. formula.

Given $T \models \mathcal{M} \supseteq \mathcal{A} \subseteq \mathcal{N} \models T$

$a = (a_1, \ldots, a_n) \in A^n$. Want $\mathcal{M} \models \varphi'(a) \iff \mathcal{N} \models \varphi'(a)$

$\mathcal{M} \models \varphi'(a) \Rightarrow \mathcal{M} \models \exists y (\bigvee \bigwedge \psi_{ij}(a,y))$

$\Rightarrow \exists i$ st in $\mathcal{M}$ $\exists$ sol to $\bigwedge_{j=1}^{\ell} \psi_{ij}(a,y)$

$\overset{(*)}{\Rightarrow} \exists i$ st in $\mathcal{N}$ "

$\Rightarrow \mathcal{N} \models \varphi'(a)$

By symmetry, $\mathcal{M} \models \varphi'(a) \iff \mathcal{N} \models \varphi'(a)$

By 6.6, $\varphi'(x)$ is T-equiv to q.f. $\therefore$ $\varphi(x)$ is T-eq to a q.f. ∎

<u>Example</u> $L = \emptyset$, T = theory of infinite sets. has QE

Proof: Use criterion. M, N infinite sets, $A \subseteq M$, $A \subseteq N$. $\psi(y)$ a conjuction of L_A-literals with a sol. in M.
Want: $\psi(y)$ has a sol in N. Possible conjuncts in $\psi(y)$:

1. $y = y$
2. $y = a$, $a \in A$
3. $y \neq y$
4. $y \neq a$, $a \in A$
5. L_A-literals not involving y

Note ③ cannot appear as $\psi(y)$ has a sol (in M). We can drop ① and get an equivalent formula as ① is true in every structure.
Conjuncts in $\psi(y)$ of the form ⑤ are true in M, so in A, and so also in N, so can drop them too.
If ② appears then $\psi(y)$ has a sol in A and so in N and we are done. So we may assume ② does not appear. So $\psi(y)$ is of the form

$$\bigwedge_{i=1}^{k} y \neq a_i \quad , \quad a_i \in A$$

This has a solution in N as N is infinite. ⊠

Corollary: $M \models T$ (as in above ex) and $X \subseteq M$ is a definable set then X is finite or cofinite (exercise)

ex DLO has QE

Proof: $(M, <), (N, <) \models$ DLO $A \subseteq M$, $A \subseteq N$
$\psi(y)$ a conjunction of L_A-literals with a sol in $(M, <)$.
Want: sol in $(N, <)$.

① $y = y$
② $y < y$
③ $y = a$
④ $y < a$
⑤ $a < y$
($a \in A$)

⑥ $y \neq y$
⑦ $y \geq y$
⑧ $y \neq a$
⑨ $y \geq a$
⑩ $a \geq y$

⑪ L_A-literals without y

"so, if, yeah, right" "irregardless"

①,⑦ true in every structure, drop them
②,⑥ <u>cannot</u> appear
⑨,⑩ are equivalent to disjunctions of the form ③,④,⑤ and so we may assume they don't appear
⑪ is true in A and so in N, so we can drop it too
if ③ appears in $\psi(y)$ then $\psi(y)$ has a sol in A, so in N
So we may assume $\psi(y)$ is of the form

$$\bigwedge_{i=1}^{r} y < a_i \wedge \bigwedge_{j=1}^{s} y > b_j \wedge \bigwedge_{k=1}^{t} y \neq c_k$$

Since $\psi(y)$ has a sol, it must be that $a_i > b_j \ \forall i,j$
Now $N \models$ DLO so it has infinitely many elements δ st $a_i > \delta > b_j \ \forall i,j$
In particular, there is a $\delta \neq c_k \ \forall k$. This is a sol of $\psi(y)$ in N. ∎

Example $L = \{0, 1, +, -, \times\}$ $T = ACF$.
$K, K' \models ACF$, $A \subseteq K$, $A \subseteq K'$
A subring of K, K'. A integral domain
$\mathbb{F} = Frac(A)$ exists and is unique
so we may assume

$$A \subseteq Frac(A) \begin{matrix} \subseteq K \\ \subseteq K' \end{matrix}$$

Taking algebraic closures, we may assume that

$$A \subseteq Frac(A) \subseteq Frac(A)^{alg} \begin{matrix} \subseteq K \\ \subseteq K' \end{matrix}$$

Given a conjunction $\psi(y)$ of L_A-literals, with sol in K. We have seen that $\psi(y)$ is of the form

$$\bigwedge_{i=1}^{k} P_i(y) = 0 \wedge \bigwedge_{j=1}^{\ell} Q_j(y) \neq 0$$

where $P_i, Q_j \in A[y] \setminus A$.
If $k > 0$ then any sol of $\psi(y)$ in K is in $Frac(A)^{alg} \subseteq L$ and we are done. ~~then~~ If $k = 0$, then L is infinite and there are finitely many roots of the Q_j to avoid.

Corollary: Every definable subset in $K \models ACF$ is Zariski constructable.
In particular in 1-spaces all definable sets are finite ~~or~~ or cofinite.
We say that ACF is <u>strongly minimal</u>.

2015 03 23c

Corollary: In 1-space in DLO the definable sets are finite unions of points and open intervals.
Such theories are called o-minimal ("oh-minimal")

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$Th(\mathbb{R},0,1,+,-,\times)$ does not admit QE: Since $<$ is definable but not q.f. definable ($(0,\infty)$).
One can show that $Th(\mathbb{R},0,1,+,-,\times,<)$ has QE (won't prove this).

Skolemisation. $\mathcal{M}$ any structure. Consider
$$\hat{L} = L \cup \{R_X \text{ n-ary relation symbol; } X \subseteq M^n \text{ definable in } \mathcal{M}\}$$
$\mathcal{M}$ into an $\hat{L}$-structure $\hat{\mathcal{M}}$, $R_X^{\hat{\mathcal{M}}} := X$.
$$Def(\hat{\mathcal{M}}) = Def(\mathcal{M})$$
$Th(\hat{\mathcal{M}})$ has QE as every definable set is atomic.

Proposition: If T has QE then it is model complete: whenever $\mathcal{M},\mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M} \preceq \mathcal{N}$.

Proof If $\varphi(x_1,\dots,x_n)$ is an L-formula, $a_1,\dots,a_n \in M$. By QE, φ is T-equivalent to ~~the~~ some $\psi(x_1,\dots,x_n)$ q.f.
$$\mathcal{M} \models \varphi(a_1,\dots,a_n) \iff \mathcal{M} \models \psi(a_1,\dots,a_n) \iff \mathcal{N} \models \psi(a_1,\dots,a_n)$$
$$\iff \mathcal{N} \models \varphi(a_1,\dots,a_n).$$
(assuming $n>0$).
If $n=0$ if σ is an L-sentence we can write $\sigma = \varphi(x)$, take $a \in M$
$\varphi(x)$ is T-equivalent to some q.f. $\psi(x)$
$$\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a) \quad \text{via previous argument}$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$\mathcal{M} \models \sigma \qquad\qquad \mathcal{N} \models \sigma$$
$\blacksquare$

Corollary: $(\mathbb{Q},<) \preceq (\mathbb{R},<)$

Proof: DLO has QE.

Corollary: $(\mathbb{Q}^{alg},0,1,+,-,\times) \preceq (\mathbb{C},0,1,+,-,\times)$

Proof ~~[illegible]~~ ~~[illegible]~~ $\varepsilon \in$ ~~[illegible]~~
ACF has

I'm not sure why
I wrote it backwards.

Application: Hilbert's Nullstellensatz:
Suppose $K \models ACF$, I proper ideal in $K[X_1, \ldots, X_n]$. Then there exists $a_1, \ldots, a_n \in K$ such that $P(a_1, \ldots, a_n)$ for all $P \in I$.

we may assume I is prime, as it is contained in a maximal (prime) ideal

Proof: $I = (P_1, \ldots, P_\ell)$, $P_1, \ldots, P_\ell \in K[X_1, \ldots, X_n]$. We are looking for a ~~finite~~ simultaneous root of $\{P_1, \ldots, P_\ell\}$.

$$K \leq K[X_1, \ldots, X_n]/I \quad \text{integral domain, embedding as } I \text{ proper}$$
$$\cap$$
$$L := \mathrm{Frac}(K[X_1, \ldots, X_n]/I)^{alg}$$

Let $a_i = X_i + I \in K[X_1, \ldots, X_n]/I \subseteq L$

$$\begin{aligned} P_i(a_1, \ldots, a_n) &= P_i(X_1 + I, \ldots, X_n + I) \\ &= P_i(X_1, \ldots, X_n) + I \\ &= 0 \qquad \text{as } P_i \in I \end{aligned}$$

We have $L \models \exists x_1 \ldots \exists x_n \left(\bigwedge_{i=1}^{\ell} P_i(x_1, \ldots, x_n) = 0\right)$, L_K-sentence
But $K \models ACF$, $L \models ACF$, $K \subseteq L$, ACF is model complete
Therefore $K \preceq L$ and so $K \models \exists x_1 \ldots \exists x_n \left(\bigwedge_{i=1}^{\ell} P_i(x_1, \ldots, x_n) = 0\right)$.
Any witness to this is a root of all $P \in I$. $\square$

Def) T an L-theory. An existentially closed model of T is a model $M \models T$ such that given any q.f.-formula $\varphi(x_1, \ldots, x_n)_M$ that has a ~~sol~~ realization in some $M \subseteq N \models T$, then M already has a realization of $\varphi(x_1, \ldots, x_n)$.

Remark: Every model of a model-complete theory is e.c.

Examples: (a) $L = \emptyset$, $T = \emptyset$. What are the e.c. models? Infinite sets.

Proof: e.c. models of T are infinite ✓
Suppose M is an inf. set. $\varphi(x, a)$ is a q.f. L_M-formula
$x = (x_1, \ldots, x_n)$, $a = (a_1, \ldots, a_\ell)$ (so $\varphi(x, y)$ is a q.f. L-formula)
$M \subseteq N$ st $N \models \exists x_1 \ldots \exists x_n \varphi(x, a)$
T_1 := theory of infinite sets. T_1 has QE, so is model complete. $M \models T_1$, $N \models T_1$, so $M \preceq N$
$\therefore M \models \exists x_1 \ldots \exists x_n \varphi(x, a)$ $\dashv$

"I'll leave the proof to you (laughs)"

2015 03 25 C

(b) $L = \{0,1,+,-,\times\}$ T = theory of integral domains
e.c. models of T are algebraically closed fields.

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Proof: Suppose R is an e.c. integral domain. For $a \in R \setminus \{0\}$, consider $\varphi(x) = ax = 1$. This L_R-formula is realized in $\mathrm{Frac}(R)$, an integral domain.
e.c. $\Rightarrow \exists$ in R, $ax = 1$ has a sol'n $\therefore$ R is a field.
Now fix $P(x) \in R[x]$, $P \neq 0$. Consider the L_R-formula $\varphi(x)$, $P(x) = 0$. $\varphi(x)$ has a realization in R^{alg}, also an int. dom
e.c. $\Rightarrow P(x) = 0$ has a sol'n in R. $\therefore$ R is an alg. closed field.
Conversely, suppose F is an alg. closed field. Let $\varphi(x,a)$ be a q.f. L_F-formula, $a \in F^m$, $x = (x_1, \dots, x_n)$, $\varphi(x,y)$ q.f. L-formula.
Suppose $\varphi(x,a)$ is realised in some int dom $F \subseteq R$.
Want: F has a realization of $\varphi(x,a)$
Say $R \models \varphi(b,a)$ for some $b \in R^n$. Let $K = \mathrm{Frac}(R)^{alg}$
so as φ is q.f., $R \subseteq K$, $K \models \varphi(b,a)$, $K \models \exists x\, \varphi(x,a)$,

$F \subseteq R \subseteq K$ $\quad\therefore F \preceq K \quad \therefore F \models \exists x\, \varphi(x,a)$

models of ACF which is model complete ☒

Remark: In previous example, we could have worked with fields instead of integral domains. ie The existentially closed fields are the algebraically closed fields.

		e.c.
$\forall\exists$ theory	sets	infinite sets
$\forall$ theories	linear orderings	DLO
	int. dom. / fields	ACF
	torsion-free abelian grps	DAG (divisible torsion-free abelian groups)

Theorem 1: Suppose T is a $\forall\exists$ theory.
(a) If $T' \supseteq T$ is such that
 (1) T' is model-complete;
 (2) every model of T embeds (as a substructure) into a model of T',
then $\mathrm{Mod}(T')$ = the e.c. models of T.
We say T' is the <u>model companion</u> of T.

<u>Def</u> We say T is <u>$\forall\exists$</u> if there exists a set Σ of L-sentences, each being

of the form
$$\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_\ell \underbrace{\varphi(x_1, \dots, x_n, y_1, \dots, y_\ell)}_{q.f.} \qquad n, \ell \geq 0$$

such that $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(T)$.
ie T is $\forall\exists$ iff it has a $\forall\exists$ axiomatization.

(b) Conversely if the class of e.c. models of T is axiomatized by some $T' \supseteq T$, then T' satisfies (1) and (2). ie T' is a model companion

What does $\forall\exists$ have to do with anything?

Proposition 2: If T is $\forall\exists$ and $(\mathcal{M}_i ; i<\alpha)$ is a <u>chain</u> of models of T (ie for $i<j$, $\mathcal{M}_i \subseteq \mathcal{M}_j$) then
$$\mathcal{M} := \bigcup_{i<\alpha} \mathcal{M}_i$$
is a model of T.

Exercise/Remark: If $(\mathcal{M}_i)$ is an elementary chain (ie for $i<j$, $\mathcal{M}_i \preceq \mathcal{M}_j$) then it is always (with without? $\forall\exists$ assumption) that $\mathcal{M} \models T$. In fact, $\mathcal{M}_i \preceq \mathcal{M}$ for all $i<\alpha$.

Note: There is a canonical L-structure $\mathcal{M} = \bigcup_i \mathcal{M}_i$.

Proof: Take an axiom $\forall x \exists y\, \varphi(x,y)$ of T. $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_\ell)$, φ is q.f.. Let $a \in M^n$ be arbitrary. Need $\mathcal{M} \models \exists y\, \varphi(a,y)$.
$M = \bigcup_i M_i$, so $a \in M_i^n$ for some i. $\mathcal{M}_i \models T$ so $\mathcal{M}_i \models \forall x \exists y\, \varphi(x,y)$
$\Rightarrow \mathcal{M}_i \models \exists y\, \varphi(a,y) \Rightarrow \mathcal{M}_i \models \varphi(a,b)$ some $b \in M^\ell$
But $\mathcal{M}_i \subseteq \mathcal{M}$ and φ is q.f. so $\mathcal{M} \models \varphi(a,b)$. So $\mathcal{M} \models \exists y\, \varphi(a,y)$.
$\therefore \mathcal{M} \models \forall x \exists y\, \varphi(x,y)$. ∎

Corollary 3: T $\forall\exists$ theory, $\mathcal{M} \models T$. Then there exists $\mathcal{N} \supseteq \mathcal{M}$ st $\mathcal{N}$ is an e.c. model of T.

Proof: Enumerate q.f. L_M-formulas

$$(\varphi_\alpha(x_\alpha) ; \alpha < \kappa) \qquad x_\alpha \text{ finite tuple of variables}$$

Recursively construct a chain $(\mathcal{M}_\alpha ; \alpha<\kappa)$ of models of T. $\mathcal{M}_0 = \mathcal{M}$.
If α limit ordinal, define

$$\mathcal{M}_\alpha = \bigcup_{i<\alpha} \mathcal{M}_i$$

By prop, $\mathcal{M}_\alpha \models T$. Else, definition of $\mathcal{M}_{\alpha+1}$:
If $\varphi_\alpha(x_\alpha)$ has a realization in some extension $\mathcal{M}' \supseteq \mathcal{M}_\alpha$ model of T, then set $\mathcal{M}_{\alpha+1} := \mathcal{M}'$, otherwise set $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha$.
Let

$$\mathcal{N}_0 = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$$

Again, by prop, $\mathcal{N}_0 \models T$. It has the property that every q.f. L_M-formulas with a realization in some extension of $\mathcal{N}_0$ has a solution in $\mathcal{N}_0$.
Now repeat the construction to build $\mathcal{N}_0 \subseteq \mathcal{N}_1 \models T$ as above but with $\mathcal{N}_0$ in place of $\mathcal{M}$. Iterate to get

$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq .$$

Let

$$\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i$$

This $\mathcal{N}$ is an e.c. model of T. ∎

Fact 4: (Converse to proposition 2)
If $\mathrm{Mod}(T)$ is closed under unions of chains then T is $\forall\exists$.

Corollary 5: If T is model complete then T is $\forall\exists$.

Proof: Suppose $(\mathcal{M}_i, i<\alpha)$ is a chain of models of T. Since T is model-complete this chain is elementary; $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ all $i<j<\alpha$. By the exercise,

$$\mathcal{M} := \bigcup_{i<\alpha} \mathcal{M}_i \models T$$

∎

Proof (of Thm (1a)): T $\forall\exists$, $T' \supseteq T$ satisfying (1), (2). We want: $\mathrm{Mod}(T')$ = the e.c. models of T.

Suppose $\mathcal{M} \models T$; let $\varphi(x)$ be a q.f. L_M-formula realized in some extension $\mathcal{N} \supseteq \mathcal{M}$, $\mathcal{N} \models T$. By (2), extend $\mathcal{N}$ to a model of T', say $\mathcal{N}'$.
Since T' is model complete (by (1)), $\mathcal{M} \preceq \mathcal{N}'$ (as $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{N}'$).
But $\mathcal{N} \models \exists x \underbrace{\varphi(x)}_{\text{q.f.}} \Rightarrow \mathcal{N}' \models \exists x \varphi(x)$.
$\Rightarrow \mathcal{M} \models \exists x \varphi(x)$. $\therefore \mathcal{M}$ is e.c.
Conversely suppose $\mathcal{M}$ is an e.c. model of T. By (2), $\mathcal{M} \subseteq \mathcal{M}' \models T'$.
By cor. 5, since (1), T' is $\forall\exists$.
Let $\forall x \exists y\, \varphi(x,y)$ be an axiom for T'.
$\mathcal{M}' \models \forall x \exists y\, \varphi(x,y)$. Let $a \in M^n$, so $\mathcal{M}' \models \exists y\, \varphi(a,y)$
ie $\varphi(a,y)$ is a q.f. L_M-formula realized in $\mathcal{N}' \supseteq \mathcal{M}$ and $\mathcal{M}' \models T$.
But $\mathcal{M}$ is e.c. $\Rightarrow$ $\varphi(a,y)$ is realized in $\mathcal{M}$. $\therefore \mathcal{M} \models \exists y\, \varphi(a,y)$.
$\therefore \mathcal{M} \models \forall x \exists y\, \varphi(x,y)$ $\therefore \mathcal{M} \models T'$. ∎

2015 04 01

We will use the following:

Lemma 6: If every model of a theory T is e.c. then T is model-complete.

Proof: We claim every existential formula $\varphi(x)$ is T-equiv. to a universal formula.
Verification: We use the criterion from A6: $\varphi(x) = \exists y\, \psi(x,y)$, ψ q.f. $x = (x_1, \dots)$, $y = (y_1, \dots)$, ℓ
Given $\mathcal{M} \subseteq \mathcal{N}$ models of T, and any $a \in M^n$, show
$\mathcal{N} \models \varphi(a) \Rightarrow \mathcal{M} \models \varphi(a)$.
So we assume $\mathcal{N} \models \exists y\, \psi(a,y)$, ie $\psi(a,y)$ has a sol'n in $\mathcal{N} \supseteq \mathcal{M}$
and $\mathcal{M}$ is e.c. so $\psi(a,y)$ has a sol'n in $\mathcal{M}$ already
$\Rightarrow \mathcal{M} \models \exists y\, \psi(a,y) \Rightarrow \mathcal{M} \models \varphi(a)$. □
So every formula is T-eq. to a universal formula
Indeed, reduce to prenex normal form:

$$Q_1 x_1 \dots Q_\ell x_\ell\, \varphi(x_1 \dots x_\ell, y_1 \dots y_n), \quad \varphi \text{ q.f.}$$

$\exists x \forall y\, \varphi(x,y)$

$\neg \forall x \neg \forall y\, \varphi(x,y)$

$\neg \forall x \underbrace{\exists y\, \varphi(x,y)}_{T\text{-equiv to some universal}}$

$\neg \forall x \forall y \dots$

$\exists x \exists y \neg \dots'$

By A6, this implies T is model complete ∎

Proof (of Theorem 1b).
Suppose T is $\forall\exists$, and $T' \supseteq T$ axiomatizes the e.c. models of T.
~~The models of T' are e.c models of T~~
Let $\mathcal{M} \models T'$. Then by assumption, $\mathcal{M}$ is an e.c. model of T. But since $T \subseteq T'$, $\mathcal{M}$ is also an e.c. model of T' (check).
By Prop 6, T' is model-complete.
If $\mathcal{M} \models T$. By Cor. 3, since T is $\forall\exists$, $\mathcal{M} \subseteq \mathcal{N}$ st $\mathcal{N}$ is an e.c. model of T. So $\mathcal{N} \models T'$ ∎

Example: The theory of groups does not have a model companion.
Note by Cor 3, every grp is a subgroup of some e.c. group.
But there is no theory of e.c. groups.

Proof. Toward a contradiction assume T' is a model companion of T. By 1a, it is theory of e.c. grps
Let G be a group st $\forall n<\omega$ $\exists a \in G$ with finite order but $\mathrm{ord}(a) > n$.
By Cor 3, $G \leq H$ where H is an e.c. group So $H \models T'$
Claim: In H, $\forall n<\omega$ $\exists \{a,b\} \in H$ st $\mathrm{ord}(a), \mathrm{ord}(b) > n$ (possibly infinite) st a, b are not conjugates
Verification. Let $a \in G$ with $\mathrm{ord}(a) = m > n$. Let $b \in G$ be st $\mathrm{ord}(b) = l > m$
Note in H, $\mathrm{ord}(a) = m$ and $\mathrm{ord}(b) = l$ (it is a q.f. fact)
Since conjugacy preserves order, and $\mathrm{ord}(a) \neq \mathrm{ord}(b)$, a and b are not conjugate.
By compactness, there is an elementary extension $H \preceq K$ st in K $\exists \alpha, \beta \in K$ of infinite order that are not conjugate
$(K$ is a model of $\mathrm{Th}(H_H) \cup \{c_1^m \neq e, m>0\} \cup \{c_2^m \neq e, m>0\}$
$\cup \{\neg\exists x (c_1 = x c_2 x^{-1})\})$

Now $K \models T'$ also. So K is an e.c group.
$\alpha = x \beta x^{-1}$
has no sol'n in K. But:

FACT: For any u,v in any group C, if u,v are of infinite order then in some ~~extens~~ group extension $C' \geq C$, u and v are conjugate

$\therefore$ that K is e.c. ∎