PMATH 733 - Model Theory and Set Theory

2015 01 05

I Set Theory

1. Language of math } our focus
   Practical set theory }
   Eg. - induction/recursion
   - size of sets (cardinality)

2. Foundations of Mathematics
   - axiomatic approach

II Model Theory

Study of (arbitrary) mathematical structures in a fixed formal language
"the theory of mathematical theories"

Chapter 1: Ordinals

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

counting: - enumerating, ordering $\rightsquigarrow$ ordinals
- measuring (finite) sets $\rightsquigarrow$ cardinals

Develop an infinitary analogue, set theoretic subtleties.
Start with some basics of posets.

1 Def. A partially ordered set (or poset) is a set E with a binary relation $R \subseteq E^2$ satisfying:

1. Reflexivity: $(a,a) \in R$ for all $a \in E$;
2. Antisymmetric: if $(a,b) \in R$ and $(b,a) \in R$ then $a=b$;
3. Transitivity: if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

A poset is linear (or total) if for all $a,b \in E$, either aRb or bRa. (Notation: we often write aRb instead of $(a,b) \in R$.)

ex A any set, $E = \mathcal{P}(A)$ = set of all subsets of A
R = containement
As long as A has more than one element, (E,R) is a poset that is not total.

ex $(\mathbb{Z}, \leq)$ is a total ordering

2 Def] A strict poset is like a poset except reflexivity is replaced by
antireflexivity: $\neg(aRa)$ for all $a \in E$
A strict poset is linear (or total) if for all $a,b \in E$, either aRb or bRa or $a=b$. ex. $(\mathbb{Z}, <)$

Remark: If (E,R) is a poset, define $aR_{\neq}b$ to mean aRb and $a \neq b$. Then $(E, R_{\neq})$ is a strict poset.
If (E,R) is a strict poset, define $aR_{=}b$ to mean aRb or $a=b$. Then $(E, R_{=})$ is a poset.

Notation: We will usually denote a poset by $(E, \leq)$ and its corresponding strict poset by $(E, <)$.

Def] A linearly ordered set $(E, \leq)$ is well-ordered if every non-empty subset of E has a least element. That is, if $D \subseteq E$ and $D \neq \emptyset$ then there exists $a \in D$ such that $a \leq b$ for all $b \in D$.

ex $(\mathbb{N}, \leq)$ is well-ordered, $(\mathbb{Z}, \leq)$ is not, neither is $(\mathbb{R}, \leq)$

3 Lemma: Well-orderings are rigid. That is, if $(E, \leq)$ is a well-ordering then the only automorphism of $(E, \leq)$ is the identity. An automorphism of a poset $(E, \leq)$ is a bijection $f: E \to E$ such that $a \leq b \iff f(a) \leq f(b)$ for all $a,b \in E$.

Proof: Suppose f is an automorphism of $(E, \leq)$. Consider 2015 01 05

$$D = \{a \in E ; f(a) \neq a\}$$

Suppose towards a contradiction, that $f \neq id$. Then $D = \emptyset$. Let $a \in D$ be least.

Case 1: $f(a) < a$. Then $f(a) \notin D$ so $f(f(a)) = f(a)$. Hence $f(a) = a$ as f is injective. Contradiction.

Case 2: $a < f(a)$. Then $f^{-1}(a) < a$ as f^{-1} is an automorphism. So $f^{-1}(a) \notin D$, so $f^{-1}(a) = a$, so $a = f(a)$. Contradiction.

By totallity, these are the only possibilities. So $f = id$. $\blacksquare$

4 Corollary: If $(E, \leq)$ and $(F, \preccurlyeq)$ are isomorphic well-orderings then there is a unique isomorphism $f : E \to F$.

An isomorphism of posets $(E, \leq)$ and $(F, \preccurlyeq)$ is a bijection $f : E \to F$ such that $a \leq b \iff f(a) \preccurlyeq f(b)$ for all $a, b \in E$.

5 Lemma: A ~~distinct~~ strict well-ordering is not isomorphic to any initial segment of itself

For a linear order $(E, <)$ and $b \in E$, let $E_{<b} := \{x \in E ; x < b\}$. Then $(E_{<b}, <)$ is the <u>initial segment of $(E, <)$ determined by b</u>

ex $((0,1), <) \xrightarrow{\cong} ((0, 1/2), <)$ via $x \mapsto x/2$

Proof: Suppose $f : (E, <) \to (E_{<b}, <)$ is an isomorphism, for $b \in E$. Let

$$D = \{x \in E ; f(x) \neq x\}$$

Note $b \in D$ since $f(b) \in E_{<b}$ and $b \notin E_{<b}$. So $D \neq \emptyset$. Let $a \in D$ be least. Break into two cases again (exercise). $\blacksquare$

We are interested in $(\mathbb{N}, <)$; it is the prototype of a strict well-ordered set. We want sets and membership to be basic and derive/construct all other math objects in terms of these.

Constructing the natural numbers in set theory

2015 01 07

Define

$$0 := \emptyset$$
$$1 := \{0\} = \{\emptyset\}$$
$$2 := \{0,1\} = \{\emptyset, \{\emptyset\}\}$$
$$3 := \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\vdots \qquad \vdots$$
$$S(n) := n \cup \{n\}$$

↑ successor function

Note: The usual ordering is induced by $\in$.

We have already implicitly been using some axioms of set theory.

Emptyset Axiom: There exists a set that has no members, denoted by $\emptyset$.

Next we need to produce $\{x\}$ from x. In fact, we introduce:

Pairset Axiom: Given sets x,y there exists a set whose elements are precisely x,y, denoted $\{x,y\}$.

Note if $x=y$ then $\{x,y\}=\{x\}$.

Extensionality: Two sets are equal if and only if they have the same members.

To get $S(n)$ from n we need:

(Notation: Given sets x,y we say x is a subset of y, $x \subseteq y$, if every element of x is an element of y.)

Unionset Axiom: Given a set x there exists a set whose members are the members of the members of x, denoted by $\bigcup x$. 2015 01 071
ie $y \in \bigcup x$ if and only if $y \in z$ for some $z \in x$

So $x \cup y := \bigcup\{x, y\}$. In particular, $S(n) = \bigcup\{n, \{n\}\}$.

While this gives us a way to produce each natural number it does not prove the existence of the "set of natural numbers".
Why not add an axiom saying: There exists a set whose members are precisely the natural numbers.
ie $y \in \omega \Leftrightarrow (y=0) \text{ or } (y=1) \text{ or} \cdots$

Problem: We want an axiom to assert the existence of a set having certain properties, where the properties are definite condition.

6 Def] $x \in y$ and $x = y$ are definite conditions, where x, y are either sets or indeterminates standing for sets.
If P, Q are definite conditions then so are:

- not P, $\neg P$
- P and Q, $P \wedge Q$
- P or Q, $P \vee Q$

(Note that $P \Rightarrow Q$ is definite since it is equivalent to $\neg P \vee Q$.)

- for all x, P, $\forall x P$
- there exists x, P, $\exists x P$

A condition is definite if it arises in this way in finitely many steps.

Note:
- Emptyset Axiom: There exists a set x satisfying
$$\neg \exists y (y \notin x) \quad ??$$
- Pairset Axiom: Given x, y, there exists a set p satisfying
$$(x \in p) \wedge (y \in p) \underset{\forall z}{\wedge} (z \in p \Rightarrow (z = x) \vee (z = y)).$$

An operation H on sets is definite if the condition $y = H(x)$ is a definite condition.

ex. The successor $S(x) = x \cup \{x\}$ is definite since

$$y = S(x) \text{ if and only if } \underset{\forall z}{\left((z \in y) \Leftrightarrow ((z \in x) \vee (z = x))\right)}$$

We will obtain the set of natural numbers as the smallest set containing 0 and closed under S.

Infinity Axiom: There exists a set I containg 0 and closed under the successor function:

$$(0 \in I) \wedge (\forall x (x \in I \Rightarrow \underbrace{S(x) \in I}_{\text{exercise: if } H(x) \text{ is a definite operation then } H(x) \in y \text{ is a definite condition}}))$$

We want to pick out the smallest such I. Suppose I is as given by the Infinity Axiom.

$$\bigcap \{J \subseteq I ; 0 \in J \text{ and if } x \in J \text{ then } S(x) \in J\}$$

We need more axioms to do this.

Powerset Axiom: Given a set x there exists a set whose elements are the subsets of x, denoted $\mathcal{P}(x)$.

$$\forall z (z \in \underbrace{\mathcal{P}(x)} \leftrightarrow \forall t (t \in z \rightarrow t \in x))$$

Separation Axiom: Given a set x and a definite condition P, there exists a set whose elements are precisely the elements of x which satisfy P, denoted by

$$\{y \in x ; P(y)\}.$$

Important that:

1. P is definite
2. we are inside a (fixed) set x

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Now, in
$$\bigcap\{J\in\mathcal{P}(I);\, 0\in J, \text{ and } x\in J \Rightarrow S(x)\in J\}$$
as before:

- I is as given by the infinity axiom
- $\mathcal{P}(I)$ is given by the powerset axiom
- the set is defined by the axiom of separation
- we can prove intersections exist by the axioms given so far (exercise) (for $X\neq\emptyset$, $y\in\bigcap X$ if and only if $y\in z$ for all $z\in X$)

7. Def] The set of <u>natural numbers</u> is the set
$$\omega := \bigcap\{J\in\mathcal{P}(I);\, 0\in J, \text{ and } x\in J\Rightarrow S(x)\in J\}$$
where I is <u>any</u> set given by the infinity axiom.

Exercise: Prove that the above is independent of choice of I.

A useful and immediate consequence of this construction of ω is:

8. <u>Induction Principle</u>: If $J\subseteq\omega$, $0\in J$, and $S(x)\in J$ whenever $x\in J$, then $J=\omega$.
Proof: We have $J\subseteq\omega$ by assumption, and we get $\omega\subseteq J$ by definition of ω. ∎

One proves a lot about ω this way.

9. Lemma: Every element of ω is a subset of ω.
Proof: Let
$$J=\{n\in\omega;\, n\subseteq\omega\}.$$
Then $0\in J$ since $0=\emptyset$ is a subset of any set vacuously. Now suppose $n\in J$. We want $S(n)\in J$. Well $S(n)=n\cup\{n\}\in\omega$. We want $S(n)\subseteq\omega$. If $t\in S(n)$ then $t=n$ or $t\in n$. If $t=n$ then $t\in\omega$, and if $t\in n$ then $t\in\omega$ as $n\subseteq\omega$. Hence $S(n)\subseteq\omega$. Thus $S(n)\in J$. Therefore $J=\omega$.

Similarly arguments yield:

10 Proposition: $(\omega, \in)$ is a strict well-ordering.
Proof: see notes, 1.13. ∎

This motivates:

11 Def] An <u>ordinal</u> is a set α such that:
i) every element of α is a subset of α;
ii) $(\alpha, \in)$ is a strict well-ordering.

2015 01 12

☆ The Zermelo-Fraenkel axioms of set theory are:
- Extensionality
- Emptyset
- Pairset
- Unionset
- Infinity
- Powerset
- Separation
- Replacement Axiom

<u>Replacement Axiom</u>: If H is a definite operation on sets and x is a set then there exists a set y which is the image of $H|_x$.
ie There exists a set y such that

$$z \in y \iff z = H(t) \text{ for some } t \in x$$

Aside: Cartesian products and functions.

<u>Def</u>] Given sets x, y, the <u>ordered pair</u> is

$$(x,y) := \{\{x\}, \{x,y\}\}.$$

Exercise: $(x,y) = (x',y') \iff x = x'$ and $y = y'$

Given X, Y sets, $x \in X$, $y \in Y$, $(x,y) \in \mathcal{P}(\mathcal{P}(X \cup Y))$.

The set of all ordered pairs exists, and is denoted by $X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$:

$$\text{Pairs}(X,Y) = \{p \in \mathcal{P}(X,Y);\ p = \{x,y\},\ x \in X,\ y \in Y\}$$

$$\text{Singletons}(X) = \{S \in \mathcal{P}(X);\ S = \{x\} \text{ some } x \in X\}$$

$$\cancel{X \times Y = \text{Pairs}(\text{Singletons}(X), \text{Pairs}(X,Y))}$$

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Cartesian products exist.

Def| Given sets X, Y, a function $f: X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that for all $x \in X$, there is a unique element of Y, which we denote by $f(x)$, such that $(x, f(x)) \in \Gamma$.

Remark We are identifying a function with its graph. Sometimes in practice we treat functions as operations.

Exercise. A function is precisely ~~an~~ definite operation restricted to a set.

Recall the definition of an ordinal.

ex Each natural number, and ω, are ordinals.

12 Lemma: Suppose α, β are ordinals. If $\alpha \subset \beta$ then $\alpha \in \beta$.

Proof: Let $D = \beta \setminus \alpha$. Note $D \neq \emptyset$. Let $d \in D$ be the least element, with respect to $(\beta, \in)$. We show $\alpha = d$.

Note $d \in \beta$ implies $d \subseteq \beta$. So we claim $\alpha = d$ as subsets of β.

Claim: $d \subseteq \alpha$.

Verification: If not, let $x \in d \setminus \alpha$. Then $x \in \beta \setminus \alpha = D$. So $x \in D$ and $x < d$. Contradicts choice of d.

Claim: $\alpha \subseteq d$.

Verification: Let $x \in \alpha$. As $x, d \in \beta$, either $x = d$ or $x < d$ or $d < x$. We cannot have $x = d$ as $d \notin \alpha$. We cannot have $d < x$,

(Wrong proof: $d < x,\ x < \alpha \Rightarrow d < \alpha \Rightarrow d \in \alpha \Rightarrow$ ※)

as $d < x$ implies $d \in x$, and $x \in \alpha$ implies $x \subseteq \alpha$ (as α is an ordinal), and so $d \in \alpha$. Contradiction.

Therefore $x < d$, so $x \in d$, as desired.

Thus $\alpha = d \in \beta$. ∎

13 Proposition:

(a) Every member of an ordinal is an ordinal.
(b) No ordinal is a member of itself.
(c) If α is an ordinal then so is its successor $S(\alpha) := \alpha \cup \{\alpha\}$.
(d) The intersection of two ordinals is an ordinal.

Proof:

(a) Let $x \in \beta$, β an ordinal. Then $x \subseteq \beta$, and so $(x, \in)$ is a strict well-ordering. Let $y \in x$. We want $y \subseteq x$. Let $z \in y$. As $x, y, z \in \beta$ and $z < y < x$, we get $z < x$. So $z \in x$. Hence $y \subseteq x$.
(b) Suppose $\alpha \in \alpha$ for an ordinal α. As α is a strict poset, $\alpha < \alpha$. This is a contradiction as $\alpha = \alpha$.
(c) Easy enough
(d) Exercise. ∎

14 Proposition:

(a) If α, β are ordinals then either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$.
(cont. later) (b) Any set of ordinals is a strict well-ordering under $\in$.

Proof:

(a) Note $\alpha \cap \beta$ is an ordinal with $\alpha \cap \beta \subseteq \alpha$ and $\alpha \cap \beta \subseteq \beta$. If both containments are proper, then $\alpha \cap \beta \in \alpha \cap \beta$ by lemma 12. So $\alpha \cap \beta = \alpha$ (wlog), so $\alpha \subseteq \beta \Rightarrow \alpha = \beta$ or $\alpha \subset \beta$ ($\Rightarrow \alpha \in \beta$ by lem. 12).
(b) Let E be a set of ordinals. Antireflexivity follows from 13(b). Linearity follows from 14(a). We show that E is well-ordered. Suppose $A \subseteq E$ with $A \neq \emptyset$. Let $\alpha \in A$.
Case 1: $\alpha \cap A \neq \emptyset$. Let $A' = \alpha \cap A$, a non-empty subset of A. Let $a \in A'$ be least of A'. Assume, for a contradiction, that there is a $b \in A$ with $b \in a$. Then $b \in \alpha$, so $b \in \alpha \cap A = A'$, contradicting that a was least in A'. Thus a is least in A.
Case 2: $\alpha \cap A = \emptyset$. Then $\forall \beta \in A$, $\beta \notin \alpha$, so either $\beta = \alpha$ or $\alpha \in \beta$. Thus α is least in A. ∎

Remark: We only use the set E because the collection of all ordinals isn't a set.

2015 01 14

Correction:

$$X \times Y = \{p \in P(P(X \cup Y));\ \exists u,v\,[u \in p \wedge v \in p \wedge \forall z(z \in p \to z = u \vee z = v)$$
$$\wedge \exists x,y\,(x \in u \wedge \forall z(z \in v \to z = x) \wedge x \in v \wedge y \in v \wedge$$
$$\forall z(z \in v \to z = x \vee z = y) \wedge x \in X \wedge y \in Y)]$$

14 Proposition (cont.)

(c) If E is a set of ordinals then $\sup E := \bigcup E$ is an ordinal.

(d) There is no set consisting of all ordinals.

Proof:

(c) $\sup E$ is a set of ordinals by 13(a). 14(b) $\Rightarrow$ $(\sup E, \in)$ is a strict well-ordering.

Let $\alpha \in \sup E$. So $\alpha \in \gamma$ for some $\gamma \in E$. So $\alpha \subseteq \gamma$ as γ is an ordinal.
If $x \in \alpha$ then $x \in \gamma \subseteq \sup E$. So $\alpha \subseteq \sup E$. Therefore $\sup E$ is an ordinal.

(d) Suppose such a set E exists. Then 14(b) $\Rightarrow$ $(E, \in)$ is a strict well-ordering. ~~And 13(b)~~ And if $\alpha \in E$ and $x \in \alpha$ then by 13(b), x is an ordinal so $x \in E$. Hence $\alpha \subseteq E$.
Therefore E is an ordinal. Therefore $E \in E$. Contradicting 13(b). ∎

Convention: Given ordinals α and β we write $\alpha < \beta$ for $\alpha \in \beta$.

15 Lemma:

(a) α ordinal then $S(\alpha) > \alpha$ and there is nothing inbetween.

(b) $E \neq \emptyset$ set of ordinals then $\sup E$ is the least upper bound of E.

(c) E set of ordinals, there exists a least ordinal not in E.

Proof:

(a) $S(\alpha) = \alpha \cup \{\alpha\}$ so $\alpha \in S(\alpha)$ so $\alpha < S(\alpha)$. If $x < S(\alpha)$ then $x = \alpha$ or $x \in \alpha \Rightarrow x \leq \alpha$.

(b) First show upper bound. $\alpha \in E$, $\sup E < \alpha \Rightarrow \sup E \in \bigcup E = \sup E$, a contradiction.
If $\alpha < \sup E$ then $\alpha \in \gamma$, $\gamma \in E \Rightarrow \alpha < \gamma \Rightarrow \alpha$ not an upper bound.

(c) Let $\alpha = S(S(\sup E))$. We claim $E \subsetneq \alpha$. $x \in E \Rightarrow x \leq \sup E$ $\Rightarrow x < S(\sup E) \Rightarrow x < S(S(\sup E))$. So $E \subseteq \alpha$.
If $E = \alpha \Rightarrow \sup E = \sup(S(S(\sup E))) = S(\sup E)$ ⚹

$\left[\sup(S(\beta)) = \beta \quad \forall \beta\right]$

(Exercise: Why $\beta = S(\sup E)$ won't do?)
Let μ be least in $\alpha \setminus E$. Suppose $x \notin E$ ordinal.
If $x < \alpha$ then $x \geq \mu$ by choice. Otherwise $x \geq \alpha > \mu$. Therefore
μ is the least ordinal not in E ∎

1.6 Transfinite induction/recursion

16 Theorem: Suppose $P(x)$ is a definite condition satisfying:
(*) — If α is an ordinal and $P(\beta)$ for all $\beta < \alpha$, then $P(\alpha)$.
then P is true of all ordinals.
Proof: (~~$P(0)$ holds vacuously~~) Suppose $P(\alpha)$ is false. Let
$$D = \{\beta \leq \alpha ; P(\beta) \text{ is false}\} \neq \emptyset.$$
Let $\alpha_0 \in D$ be least. If $\beta < \alpha_0$ then $\beta \notin D$ so $P(\beta)$ is true.
By (*), $P(\alpha_0)$ is true. Contradiction. ∎

Def) A <u>successor ordinal</u> is an ordinal of the form $S(\alpha)$ for some α.
A <u>limit ordinal</u> is an ordinal that is not a successor.

A reformulation:

17 Corollary: (Second form of transfinite induction)
$P(x)$ definite condition such that:
1. $P(0)$ is true;
2. If $P(\beta)$ then $P(S(\beta))$;
3. If $\alpha > 0$ is a limit ordinal and $P(\beta)$ for all $\beta < \alpha$ then $P(\alpha)$.

Then P is true of all ordinals.

Transfinite Recursion

Induction is used to prove definite statements about all ordinals.
Recursion is used to construct definite operations on ordinals.

A <u>partial function</u> on ordinals is a function whose domain is a set of ordinals.

Note: If F is a definite operation on ordinals and α is an ordinal then $F|\alpha$ is a partial function on ordinals.

12 Theorem: Suppose G is a definite operation on partial functions on ordinals. Then there exists a unique definite operation on ordinals, F, satisfying

$$F(\alpha) = G(F\restriction\alpha)$$

Proof (sketch):

Second Form of transfinite recursion

typo in ...ks: need definite

19 Corollary: Suppose G_1 is a set, G_2 is a definite operation on sets, and G_3 is a definite operation on partial functions on ordinals. Then there is a definite operation ~~ø~~ F on ordinals satisfying:

$$F(0) = G_1$$
$$F(S(\alpha)) = G_2(F(\alpha)) \quad \text{for all } \alpha$$
$$F(\beta) = G_3(F\restriction\beta) \quad \text{for all limit } \beta$$

Ordinal Arithmetic

20 Def^n (Ordinal addition)
Fix an ordinal β, and define $\beta+\alpha$ for all ordinals α by transfinite recursion (2^nd form):

$$\beta+0 := \beta$$
$$\beta+S(\alpha) := S(\beta+\alpha)$$
$$0 \neq \alpha \text{ limit}, \quad \beta+\alpha = \sup\{\beta+\gamma \,;\, \gamma<\alpha\}$$

Note: In terms of transfinite recursion:

$$G_1 = \beta$$
$$G_2 = S$$
$$G_3(f) = \sup\{im(f)\}$$

Remarks:
(a) $\beta+1 = S(\beta)$
(b) not commutative: $1+\omega = \sup\{1+n \,;\, n<\omega\} = \omega \neq S(\omega) = \omega+1$
(c) ordinal arithmetic restricted to finite ordinals is usual arithmetic

21 Def^n (Ordinal Product)
Fix ordinal β, define $\beta\cdot\alpha$ for all α by recursion:

$$\beta\cdot 0 = 0$$
$$\beta\cdot S(\alpha) = \beta\cdot\alpha+\beta$$
$$0\neq\alpha \text{ limit} \quad \beta\cdot\alpha = \sup\{\beta\cdot\gamma \,;\, \gamma<\alpha\}$$

2015 01 19 (2

Remark: Not commutative

$2\cdot\omega = \sup\{2\cdot n \,;\, n<\omega\} = \omega$

$\omega\cdot 2 = \omega\cdot 1+\omega = (\omega\cdot 0+\omega)+\omega = (0+\omega)+\omega$

$= (\sup\{0+n \,;\, n<\omega\})+\omega = \omega+\omega = \sup\{\omega+n \,;\, n<\omega\} \geq \omega+1>\omega.$

22 Proposition: α,β,δ ordinals.

(a) $\alpha<\beta \iff \delta+\alpha<\delta+\beta$

(b) $\delta+\alpha=\delta+\beta \Longrightarrow \alpha=\beta$

(c) $(\alpha+\beta)+\delta = \alpha+(\beta+\delta)$

(d) for all $\delta\neq 0$, $\alpha<\beta \iff \delta\cdot\alpha<\delta\cdot\beta$

(e) for all $\delta\neq 0$, $\delta\cdot\alpha=\delta\cdot\beta \Longrightarrow \alpha=\beta$.

(f) $(\alpha\cdot\beta)\cdot\delta = \alpha\cdot(\beta\cdot\delta)$

23 <u>Def</u> (ordinal exponentiation)

$$\beta^0 := 1$$
$$\beta^{S(\alpha)} := \beta^\alpha\cdot\beta$$
$$0\neq\alpha \text{ limit} \qquad \beta^\alpha := \sup\{\beta^\gamma \,;\, \gamma<\alpha\}$$

24 Theorem. Every strict well-ordering is isomorphic to an ordinal. Moreover, both the ordinal and the isomorphism are unique.

Proof: Suppose $(E,\prec)$ is a strict well-ordering. Uniqueness of the isomorphism follows from corollary 4 (rigidity). Next we show uniqueness of the ordinal. Note for ordinals α,β, $\alpha<\beta$ if and only if α is an initial segment of β. Now suppose $(E,\prec)$ was isomorphic to both α and β. Then either $\alpha<\beta$ or $\alpha=\beta$ or $\beta<\alpha$. If $\alpha<\beta$ then since $f:E\to\beta$ is an isomorphism, and α is an initial segment of β, we get an initial segment $E_{f^{-1}(\alpha)}$ of $(E,\prec)$. So

$$E_{f^{-1}(\alpha)} \cong \alpha \cong E.$$

This contradicts lemma 5, as no well-ordering is isomorphic to an initial segment of itself. Similarly we cannot have $\beta<\alpha$, so $\alpha=\beta$.

Finally we show existence. Let

$$A = \{x\in E \,;\, (E_x,\prec) \text{ is isomorphic to an ordinal}\}.$$

Note if $E=\emptyset$ then $\alpha=0$ works. So assume $E\neq\emptyset$. Let $x\in E$ be least wrt $\prec$. Then $E_x=\emptyset=0$ so $x\in A$. Hence $A\neq\emptyset$.

Now define f on A by letting $f(a)$ be the ordinal which is isomorphic to $(E_a, \prec)$. Then $\text{im}(f)$ is a set of ordinals, so let α be the least ordinal not in $\text{im}(f)$.

We claim:

1) $\alpha = \text{im}(f)$, so $f: A \to \alpha$ is surjective;
2) $f: A \to \alpha$ is injective;
3) $f: (A, \prec) \xrightarrow{\cong} (\alpha, \in)$;
4) $E = A$.

Exercise (or see notes). Thus $(E, \prec) = (A, \prec) \cong (\alpha, \in)$. ∎

2015 01 21

Ordinals are a good transfinite extension of ω for enumeration but not for measuring size: $\omega+1 > \omega$ have the same size.

<u>Def</u> Two sets A, B are <u>equinumerous</u>, denoted $|A| = |B|$, if there is a bijection $f: A \to B$.

Note: We have not yet defined $|A|$.

Proposition (Schröder-Bernstein)
$|A| = |B|$ if and only if there exist injections $f: A \to B$ and $g: B \to A$.

Lemma: For any infinite ordinal α, $|\alpha| = |\alpha+1|$.
Proof: Define $g: \alpha+1 \to \alpha$ by

$$g(x) = \begin{cases} x+1 & \text{if } x < \omega \\ 0 & \text{if } x = \alpha \\ x & \text{otherwise} \end{cases}$$

<u>Def</u> A <u>cardinal</u> is an ordinal that is not equinumerous to any (strictly) lesser ordinal.

Examples:

- $n < \omega$, n is a cardinal
- ω is a cardinal

Note: Every infinite cardinal is a limit ordinal. (By the lemma.) The converse is false. Ex $|\omega+\omega| = |\omega|$.

<u>Def</u> A set is countable if it is equinumerous with ω or some $n<\omega$.

Proposition. For every set E there is a unique cardinal, denoted $h(E)$, which is the least ordinal that is <u>not</u> equinumerous to any subset of E.

Note: $h(\omega)$ is an uncountable cardinal.

Proof: Suppose such an ordinal exists, $h(E)$, then $h(E)$ is cardinal: Indeed, if $\beta < h(E)$ then $|\beta| = |A|$, some $A \subseteq E$. Hence $|h(E)| \neq \beta$. Consider

$$W := \{(A,\prec)\ ;\ A \subseteq E,\ \prec \text{ well-ordering on } A\}$$

$$\cap\ P(E) \times P(E \times E)$$

W is a set by separation. If $(A,\prec) \in W$, theorem 1.36 says that $(A,\prec) \cong (\alpha,\in)$. Write $\alpha = f(A,\prec)$. We have a definite operation. By replacement, $\mathrm{Im}(f)$ is a set.
We claim that $\mathrm{Im}(f) = \{\alpha\ ;\ \alpha = |A|, \text{ some } A \subseteq E\}$.

<sketch, see notes>

So $h(f)$ least ordinal not in $\mathrm{Im}(f)$ works. Uniqueness is easy. ∎

We have lots of uncountable cardinals. We really want every set to be equinumerous with a cardinal. This would imply every set admits a strict well-orderings. This cannot be proved from ZF. We need a 9^{th} axiom.

<u>Def</u> Suppose $\mathcal{F}$ is a set (of sets). A <u>choice function</u> on $\mathcal{F}$ is a function $c : \mathcal{F} \to \bigcup \mathcal{F}$ such that $c(F) \in F$ for all $F \in \mathcal{F}$.

<u>Axiom of choice</u>: Every set of non-empty sets has a choice function

Remarks:
(1) This too is a set existence axiom:
"There exists a subset $\Gamma \subseteq \mathcal{F} \times \bigcup \mathcal{F}$ such that for all $F \in \mathcal{F}$ there is a unique $d \in \bigcup \mathcal{F}$ such that $(F, d) \in \Gamma$, and such that $d \in F$."
(2) It is <u>not</u> the case that whenever you choose elements from sets, you are necessarily using the axiom of choice.

ex Suppose $\mathcal{F}$ is a set of non-empty subsets of ω. Then $c: \mathcal{F} \to \bigcup \mathcal{F}$ given by $c(F)$ being the least of F is a choice function. We didn't use choice.

ex Suppose $\mathcal{F} = \{A\}$ for $A \neq \emptyset$. Let $x \in A$. Then $c: \mathcal{F} \to \bigcup \mathcal{F}$ given by $c(F) = x$ is a choice function. We did not use choice.

Theorem: The following are equivalent:
1) Axiom of choice; (AC)
2) Well Ordering Principle; (WOP) (every set admits a strict well-ordering)
3) Zorn's Lemma. (if (E, R) is a strict poset in which every totally ordered subset ("chain") has an upper bound, then (E, R) has a maximal element)

Proof: (1)$\Rightarrow$(2): Let A be a set. Let $c: \mathcal{P}(A) \setminus \{\emptyset\} \to \bigcup \mathcal{P}(A) = A$ be a choice function. Define F recursively on ordinals to A by
$$F(\alpha) = \begin{cases} c(A \setminus \mathrm{Im}(F \restriction \alpha)) & \text{if } A \setminus \mathrm{Im}(F \restriction \alpha) \neq \emptyset, \\ \Theta & \text{otherwise,} \end{cases}$$
where Θ is an ordinal not in A.
We claim that $F(\alpha) = \Theta$ for some ordinal $\alpha < h(A)$.
~~Assume, for a contradiction, that F doesn't halt~~
Note that if $F(\alpha) \neq \Theta$ $\forall \alpha \in E$ for some set E of ordinals, then $F \restriction E: E \to A$ is injective. So if the claim fails, then $F \restriction h(A): h(A) \to A$ is an injection. Contradiction.
So let α be the least ordinal such that $F(\alpha) = \Theta$. So we have

$F\restriction\alpha . \alpha \hookrightarrow A$. As $F(\alpha) = \Theta$, $\mathrm{Im}(F\restriction\alpha) = A$. Thus $F\restriction\alpha$ is a bijection. We can now use this bijection and the well-ordering on α to get a well-ordering on A.

2015 01 26

(2)⇒(3): Suppose (E,R) is a non-empty strict poset satisfying

(*) — every chain has an upper bound.

Assume, for a contradiction, that (E,R) has no maximal element. Let $<$ be a strict well-ordering on E. Let $h(E)$ be the least ordinal not equinumerous with any subset of E. Build an embedding of $h(E)$ into E, by recursion:

$$F : h(E) \to E$$

$$F(0) = e \quad \text{(some fixed element of } E$$

any $\alpha < h(E)$: $F(\alpha+1) = <$-least element $x \in E$ such that $F(\alpha) R x$

(possible as $F(\alpha)$ is not maximal in E)

$0 < \beta < h(E)$ limit: $F(\beta) = <$-least $x \in E$ st $F(\gamma) R x \ \forall \gamma < \beta$

(possible since $\mathrm{Im}(F\restriction\beta)$ is a chain in E) (by (*))

Thus F is injective.

(3)⇒(1): Let $\mathcal{F}$ be a set of non-empty sets, Want a choice function on $\mathcal{F}$. Let

$$\Lambda = \{\text{all partial choice functions}\}$$
$$= \{ f : \tilde{G} \to \bigcup \tilde{G} \,;\, f(G) \in G \ \forall G \in \tilde{G},\ \tilde{G} \subseteq \mathcal{F} \}$$

Note $\Lambda \neq \emptyset$ as $\emptyset \in \Lambda$. Consider the poset $(\Lambda, \subset)$ - identifying functions with their graphs. Then (*) is true since if I is a chain in $(\Lambda, \subset)$ then $\bigcup I \in \Lambda$ is an upper bound for I.

Zorn's lemma implies that there is a maximal $f : \tilde{G} \to \bigcup \tilde{G}$ choice function, $\tilde{G} \subseteq \mathcal{F}$. If $\tilde{G} \subsetneq \mathcal{F}$, let $F \in \mathcal{F} \setminus \tilde{G}$, $x \in F$. Then

$$f \cup \{(F, x)\}$$

is a partial choice function strictly extending f, contradicting maximality. So $\tilde{G} = \mathcal{F}$ and f is a choice function on $\mathcal{F}$. ∎

We work in ZFC from now on. (9 axioms)

Lemma: Every set is equinumerous with a unique cardinal.
Proof: Uniqueness is clear: distinct cardinals are not equinumerous.
Existence: Let X be a set. Let $<$ be a strict well-ordering on X. Then $1.36 \Rightarrow (X,<) \simeq (\alpha, \in)$ for some ordinal α. In particular, $|X| = |\alpha|$. Let
$$S = \{\beta \leq \alpha ; |\beta| = |X|\}.$$
Let $\beta \in S$ be least. Then β is a cardinal: if $\gamma < \beta$ and $|\beta| = |\gamma|$ then $|\gamma| = |X|$ so $\gamma \in S$, contradicting minimality of β. Then β is as desired. ▨

Denote the unique cardinal equinumerous with X by $|X|$, and call it the <u>cardinality</u> of X.

Lemma: Given A, B, we have $|A| \leq |B|$ if and only if there is an injection $A \hookrightarrow B$.
Proof: If $|A| \leq |B|$ then $A \hookrightarrow\!\!\!\rightarrow |A| \subseteq |B| \hookrightarrow\!\!\!\rightarrow B$ and therefore $A \hookrightarrow B$.
Conversely, suppose $A \hookrightarrow B$. Then
$$|A| \hookrightarrow\!\!\!\rightarrow A \hookrightarrow B \hookrightarrow\!\!\!\rightarrow |B|$$
and so $|A| \hookrightarrow |B|$. If $|B| < |A|$ then $|B| \xrightarrow{\subseteq} |A|$. So S-B implies $|A| = |B|$. This contradicts that $|A|$ is a cardinal. Thus $|A| \leq |B|$. ▨

Corollary: Given A, B, there exists an embedding $A \hookrightarrow B$ or an embedding $B \hookrightarrow A$.

$|\vec{f}(A)| \leq |A|$

Lemma: Suppose $f: A \to B$ is a function. Then $|\mathrm{Im}(f)| \leq |A|$
Proof: Let
$$\mathcal{F} = \{f^{-1}(b) ; b \in \mathrm{Im}(f)\}. \qquad f^{-1}(b) \text{ means } \overleftarrow{f}(\{b\})$$
Then $\mathcal{F}$ exists by replacement as $b \mapsto f^{-1}(b)$ is definite, and is a set of non-empty subsets of A. By AC, there is a choice function c on $\mathcal{F}$. Let $g: \mathrm{Im}(f) \to A$ be given by
$$g(b) = c(f^{-1}(b)).$$
Then g is injective. ▨

2015 01 26(

Recall that X is countable means $|X| \leq \omega$.

Lemma: A countable union of countable sets is countable.
Proof: Suppose $|A| \leq \omega$ and for each $a \in A$, $|a| \leq \omega$. For $x \in A$, let $S_x = \{a \in A \,;\, x \in a\}$. Note $S_x \neq \emptyset$, and $x \longmapsto S_x$ is definite. So let $\mathcal{S} = \{S_x \,;\, x \in \bigcup A\}$. We can get a choice function $c: \mathcal{S} \to A$ (since $\bigcup \mathcal{S} = A$).
For each $a \in A$, there is an embedding $a \hookrightarrow \omega$. As before, using replacement and choice, we get for each $a \in A$ an inj. $f_a : a \to \omega$. We also have $g: A \hookrightarrow \omega$. Now define $F: \bigcup A \to \omega \times \omega$ by $F(x) = (g(c(x)), f_{c(x)}(x))$. Then F is injective (check), so $|\bigcup A| \leq |\omega \times \omega|$.
It remains to show that $|\omega \times \omega| \leq |\omega|$. But we know how to do this (exercise). ■

2015 01 28

Ordinal-enumeration of cardinals

Remark: κ cardinal, $h(\kappa)$ is the least ordinal not equinumerous with a subset of κ, is a cardinal.
$h(\kappa) > \kappa$ since $h(\kappa) = |h(\kappa)| > |\kappa| = \kappa$
If λ cardinal, $\lambda > \kappa$ then $|\lambda| = \lambda > \kappa = |\kappa|$, so λ is not equinumerous with any subset of κ. Hence $\lambda \geq h(\kappa)$.
So $h(\kappa)$ is the least ordinal greater than κ, we denote it by κ^+.
Note $\underset{\text{next cardinal}}{\kappa^+} \neq \underset{\text{next ordinal}}{\kappa + 1}$

We construct recursively the following sequence of infinite cardinals:

$$\aleph_0 := \omega$$

α ordinal $\qquad \aleph_{\alpha+1} := \aleph_\alpha^+$

$\beta > 0$ limit $\qquad \aleph_\beta := \sup\{\aleph_\gamma \,;\, \gamma < \beta\}$

Lemma: For all α, $\aleph_\alpha$ is an infinite cardinal.
Proof: By induction. Limit stage: suppose $\beta > 0$ is a limit ordinal

Suppose $\alpha < \aleph_\beta$ ordinal. Hence $\alpha < \aleph_\gamma$ for some $\gamma < \beta$. By induction, it is an infinite cardinal. Hence $|\alpha| < |\aleph_\gamma| \le |\aleph_\beta|$. Thus $\aleph_\beta$ is a cardinal. ■

Lemma: If $\alpha < \beta$ then $\aleph_\alpha < \aleph_\beta$.
Proof: Exercise: induction. ■

So there is a strictly increasing ordinal-enumeration of cardinals.

Lemma: For all ordinals α, $\alpha \le \aleph_\alpha$.
Proof: Induction. Limit stage: $\beta > 0$ limit ordinal. For all $\alpha \in \beta$, $\alpha \le \aleph_\alpha < \aleph_\beta$, so $\beta = \sup \beta \le \aleph_\beta$ ■

Note: Equality is possible in this lemma, though only for limit ordinals.

Proposition: Every infinite cardinal is of the form $\aleph_\alpha$, for some ordinal α.
Proof: Suppose κ is an ~~infinite~~ cardinal.

$$\kappa \le \aleph_\kappa < \aleph_{\kappa+1}$$

So it suffices to show by induction on ordinals,

(*) For every ordinal β, and every infinite cardinal $\kappa < \aleph_\beta$, there is an ordinal $\alpha < \beta$ such that $\kappa = \aleph_\alpha$.

$\beta = 0$: ✓

$\beta + 1$: Suppose $\kappa < \aleph_{\beta+1} = \aleph_\beta^+$. So either $\kappa = \aleph_\beta$ ✓ or $\kappa < \aleph_\beta$ ✓ by induction.

β limit: $\kappa < \aleph_\beta = \sup\{\aleph_\gamma ; \gamma < \beta\}$, so $\kappa < \aleph_\gamma$ for some $\gamma < \beta$ ✓ by induction. ■

Def] A <u>successor cardinal</u> is one of the form $\aleph_{\alpha+1}$ for some ordinal α. A <u>limit cardinal</u> is one of the form $\aleph_\beta$ for some limit ordinal β or 0.

Note: All (infinite) cardinals are limit ordinals.
And κ is a successor cardinal if and only if $\kappa = \lambda^+$ for some cardinal λ.

Theorem (Cantor's Diagonalization Theorem)
For any set E, $|E| < |\mathbb{P}(E)|$.
Proof: $E \hookrightarrow \mathbb{P}(E)$, $e \mapsto \{e\}$ implies $|E| \leq |\mathbb{P}(E)|$.
Suppose $|E| = |\mathbb{P}(E)|$, say $E \xrightarrow{f} \mathbb{P}(E)$. Let
$$\Delta = \{e \in E ; e \notin f(e)\} \in \mathbb{P}(E).$$
So $\Delta = f(e)$ for some $e \in E$. Then $e \in \Delta$ if and only if $e \notin \Delta$.
Contradiction. ∎

The <u>Continuum Hypothesis</u> says that $\aleph_1 = |\mathbb{P}(\aleph_0)|$.
(Equivalently, every subset of $\mathbb{R}$ is either countable or of size $|\mathbb{R}|$.)

The <u>generalized Continuum Hypothesis</u> says that $\kappa^+ = |\mathbb{P}(\kappa)|$ for all infinite cardinals κ.

These are independent of ZFC. We usually do not assume either, nor their negations.

<u>Cardinal Arithmetic</u>

<u>Def</u> κ_1, κ_2 ordinals. The <u>cardinal sum</u>
$$\kappa_1 + \kappa_2 := |X_1 \cup X_2|$$
where $X_1 \cap X_2 = \emptyset$, $|X_1| = \kappa_1$, $|X_2| = \kappa_2$.

Note: This does not depend on the choice of X_1, X_2.

The <u>cardinal product</u>
$$\kappa_1 \cdot \kappa_2 := |X_1 \times X_2|$$
where $|X_1| = \kappa_1$, $|X_2| = \kappa_2$.

Again, this does not depend on the choice of X_1, X_2.

Note: Cardinal sum and product are <u>not</u> the same as ordinal sum and product.

Properties:

(a) Cardinal sum/product are commutative and associative

(b) $\kappa_1\cdot(\kappa_2+\kappa_3)=\kappa_1\cdot\kappa_2+\kappa_1\kappa_3$

(c) $0+\kappa=\kappa$, $0\cdot\kappa=0$ $\forall\kappa$

Proof: Exercise. ∎

Theorem: Suppose κ is an infinite cardinal. Then $\kappa\cdot\kappa=\kappa$.

Proof: $\kappa\hookrightarrow\kappa\times\kappa$, $x\mapsto(x,x)$, so $\kappa=|\kappa|\le|\kappa\times\kappa|=\kappa\cdot\kappa$.

For the converse, for any ordinal α, define on $\aleph_\alpha\times\aleph_\alpha$ an ordering $<$ as follows:

$(x_1,y_1)<(x_2,y_2)$ if $\max(x_1,y_1)<\max(x_2,y_2)$

or $\max(x_1,y_1)=\max(x_2,y_2)$ and $x_1<x_2$

or " and $x_1=x_2$ and $y_1<y_2$

Claim: $(\aleph_\alpha\times\aleph_\alpha,<)$ is a strict well-ordering (see notes).

Claim: Every initial segment of $(\aleph_\alpha\times\aleph_\alpha,<)$ is of size less than $\aleph_\alpha$.

Verification: We proceed by induction on α. Easy if $\alpha=0$.
Suppose $\alpha>0$ and S is the initial segment for (x_0,y_0).
Let $z=\max\{x_0,y_0\}+1$, so that $S\subseteq z\times z$. Now $|z|\le z<\aleph_\alpha$, so $z=\aleph_\beta$ for some $\beta<\alpha$. Hence

$$|S|\le|z\times z|=|z|\cdot|z|=\aleph_\beta\cdot\aleph_\beta=\aleph_\beta<\aleph_\alpha$$

by inductive assumption.

Now suppose $\kappa=\aleph_\alpha$ and $\aleph_\alpha<\aleph_\alpha\cdot\aleph_\alpha$. By the first claim, $(\aleph_\alpha\times\aleph_\alpha,<)$ is a strict well-ordering, and so is isomorphic to $(\gamma,\in)$ for some ordinal γ. Now

$$\aleph_\alpha<\aleph_\alpha\cdot\aleph_\alpha=|\aleph_\alpha\times\aleph_\alpha|=|\gamma|\le\gamma$$

so $\aleph_\alpha$ is an initial segment of γ. So $f^{-1}(\aleph_\alpha)$ is an initial segment of $(\aleph_\alpha\times\aleph_\alpha,<)$. The second claim says $|f^{-1}(\aleph_\alpha)|<\aleph_\alpha$. But f is a bijection. Contradiction. Thus $\aleph_\alpha\ge\aleph_\alpha\cdot\aleph_\alpha$. ∎

2015 02 1

Corollary: κ_1,κ_2 non-zero cardinals, not both finite. Then

$$\kappa_1+\kappa_2=\kappa_1\cdot\kappa_2=\max\{\kappa_1,\kappa_2\}.$$

Proof: Suppose $\kappa_1 \le \kappa_2$, so κ_2 is infinite. Since

$$\begin{aligned} \kappa_2 &\hookrightarrow \kappa_1 \times \kappa_2 \\ x &\longmapsto (0, x) \end{aligned}$$

we have $\kappa_2 \le \kappa_1 \cdot \kappa_2$. But $\kappa_1 \times \kappa_2 \subseteq \kappa_2 \times \kappa_2$ so $\kappa_1 \cdot \kappa_2 \le \kappa_2 \cdot \kappa_2$. So by theorem, all are equality, so $\kappa_1 \cdot \kappa_2 = \kappa_2 = \max\{\kappa_1, \kappa_2\}$.

Next, ~~$\kappa_2 \le \kappa_1 + \kappa_2 \le \kappa_2 + \kappa_2 \le \kappa_2$, so equality holds and $\kappa_1 + \kappa_2 = \kappa_2 = \max\{\kappa_1, \kappa_2\}$~~

$$\begin{aligned} \kappa_2 &\le \kappa_1 + \kappa_2 && (\quad \downarrow \quad) \\ &\le \kappa_2 + \kappa_2 && (\kappa \le \lambda \Rightarrow \kappa + \gamma \le \lambda + \gamma) \\ &= 2 \cdot \kappa_2 && (\text{bij. } 2 \times \kappa_2 \leftrightarrow X \cup Y,\ X \cap Y = \emptyset,\ |X| = |Y| = \kappa_2) \\ &= \max\{2, \kappa_2\} && (\text{above}) \\ &= \kappa_2 && (\kappa_2 \text{ infinite}) \end{aligned}$$

So equality holds and $\kappa_1 + \kappa_2 = \kappa_2 = \max\{\kappa_1, \kappa_2\}$. ∎

Def] Given sets I and X, an <u>I-sequence in X</u> is simply a function $f: I \to X$, often denoted by $(a_i ; i \in I)$, where $a_i = f(i)$. Suppose $(X_i ; i \in I)$ is an I-sequence of sets (in some X). Then the <u>Cartesian Product</u> of $(X_i ; i \in I)$ is

$$\bigtimes_{i \in I} X_i,$$

the set of all I-sequences $(a_i ; i \in I)$ in $\bigcup X$ such that $a_i \in X_i$.

Note: If I and each X_i are non-empty then by AC, $\bigtimes_{i \in I} X_i \ne \emptyset$.

Def] Suppose $(\kappa_i ; i \in I)$ is a sequence of cardinals. Then

$$\sum_{i \in I} \kappa_i := \left| \bigcup_{i \in I} Y_i \right|$$

where $(Y_i ; i \in I)$ is a sequence of sets such that $|Y_i| = \kappa_i$ and $Y_i \cap Y_j = \emptyset$ if $i \ne j$, and

$$\prod_{i \in I} \kappa_i := \left| \bigtimes_{i \in I} Y_i \right|$$

where $(Y_i ; i \in I)$ is a sequence of sets with $|Y_i| = \kappa_i$.

Well-defined: Suppose $(Y_i' ; i \in I)$ (pairwise disj.), $|Y_i'| = \kappa_i$. Using AC, we get a sequence of bijections $(f_i ; i \in I)$, $f_i : Y_i' \to Y_i$. Now

$$\bigcup_{i\in I} Y_i' \longrightarrow \bigcup_{i\in I} Y_i$$
$$y \longmapsto f_i(y) \quad \text{where } y \in Y_i'.$$
$$\bigtimes_{i\in I} Y_i' \longrightarrow \bigtimes_{i\in I} Y_i$$
$$(a_i;\, i\in I) \longmapsto (f_i(a_i);\, i\in I)$$

Infinite sums also trivialize:

Proposition. If λ is an infinite cardinal and $(\kappa_i;\, i<\lambda)$ is a sequence of non-zero cardinals then
$$\sum_{i<\lambda} \kappa_i = \max\{\lambda, \underbrace{\sup\{\kappa_i;\, i<\lambda\}}_{\text{is a cardinal}}\}$$

Proof: Let
$$\kappa = \sup\{\kappa_i;\, i<\lambda\}.$$
Note ~~(check)~~
$$\begin{aligned}
\sum_{i<\lambda} \kappa_i &\le \sum_{i<\lambda} \kappa && \text{(check)}\\
&= \lambda\cdot\kappa && \text{(check - similar to } 2\cdot\kappa = \kappa+\kappa)\\
&= \max\{\lambda, \kappa\}. && \text{(before)}
\end{aligned}$$
Conversely,
$$\begin{aligned}
\lambda &= \sum_{i<\lambda} 1 && \text{(as above with } \kappa=1\text{, plus } \lambda\cdot 1 = \lambda)\\
&\le \sum_{i<\lambda} \kappa_i,\\
\kappa &= \sup\{\kappa_i;\, i<\lambda\} && \text{(def}^n\ \kappa)\\
&= \bigcup_{i<\lambda} \kappa_i && \text{(def sup)}\\
&\le \sum_{i<\lambda} \kappa_i, && \text{(exercise)}
\end{aligned}$$
so
$$\max\{\lambda,\kappa\} = \lambda\cdot\kappa \le \Big(\sum_{i<\lambda}\kappa_i\Big)\Big(\sum_{i<\lambda}\kappa_i\Big) = \sum_{i<\lambda}\kappa_i. \quad \blacksquare$$

What about infinite products?

Example: λ ~~is~~ infinite cardinal, $\kappa_i = 2$ $\forall i < \lambda$.

$$\prod_{i<\lambda} 2 = \left| \bigtimes_{i<\lambda} 2 \right| = |\mathcal{P}(\lambda)| > \lambda, 2$$

$$(a_i;\, i<\lambda) \longmapsto \{i<\lambda;\, a_i = 1\}$$

<u>Def</u>] κ, λ cardinals

$$\kappa^\lambda := |\{\text{functions from } \lambda \text{ to } \kappa\}|$$

$$= \prod_{i<\lambda} \kappa.$$

Note: $2^\lambda = |\mathcal{P}(\lambda)| > \lambda$

Exercise

λ ~~κ~~ $\leq \mu \Rightarrow \kappa^\lambda \leq \kappa^\mu$

$\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu$

$(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$

Theorem (König's Theorem):

$(\kappa_i;\, i\in I)$, $(\lambda_i;\, i \in I)$ sequences of cardinals, $\kappa_i < \lambda_i$ $\forall i \in I$

Then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Note: $\sum_{i<\lambda} 1 = \lambda > 1 = \prod_{i<\lambda} 1.$

Proof: $(X_i;\, i\in I)$ pairwise disjoint, $|X_i| = \kappa_i$.

$(Y_i;\, i\in I)$, $|Y_i| = \lambda_i$.

$f_i : X_i \hookrightarrow Y_i$ injective (<u>not</u> surjective)

Construct

~~Construct~~ $\bigcup_{i\in I} X_i \xhookrightarrow{f} \bigtimes_{i\in I} Y_i$ choose $c_i \in Y_i \setminus f_i(X_i)$ for each $i \in I$

$x \longmapsto f(x)$ $\quad f(x)_i = f_i(x) \quad x \in X_i$

$\qquad\qquad\qquad f(x)_j = c_j \quad x \notin X_j$

Check f is injective. So

$$\sum_{i\in I} \kappa_i \le \prod_{i\in I} \lambda_i .$$

Suppose toward a contradiction that there is a surjective map

$$h : \bigcup_{i\in I} X_i \longrightarrow \bigtimes_{i\in I} Y_i$$

For each ~~Fix~~ $i_0 \in I$. Coordinate projection

$$\bigtimes_{i\in I} Y_i \xrightarrow{\pi_{i_0}} Y_{i_0}$$

is surjective by AC, ($Y_i \neq \emptyset$ as $\lambda_i > \kappa_i \geq 0$). Hence ~~the~~ $\pi_{i_0} \circ h$ is surjective. ~~Let~~

$$\cancel{a = (a_i, i \in I) \in \bigtimes_{i\in I} Y_i}$$

Consider

$$X_{i_0} \xrightarrow{\pi_{i_0} \circ h \restriction X_{i_0}} Y_{i_0}$$

which is not surjective as $\kappa_{i_0} < \lambda_{i_0}$. Choose $c \in Y_{i_0} \setminus X_{i_0}$.
Let

$$a \in \bigtimes_{i\in I} Y_i$$

such that $a_{i_0} = c$. Let

$$x \in \bigcup_{i\in I} X_i$$

such that $\pi_{i_0}(h(x)) = a$

For each $j \in I$,

$$\overbrace{X_j \hookrightarrow \bigcup_{i\in I} X_i \xrightarrow{h} \bigtimes_{i\in I} Y_i \xrightarrow{\pi_j} Y_j}^{h_j}$$

Note h_j is not surjective as $\kappa_j < \lambda_j$. Let

$$c \in \bigtimes_{i\in I} Y_i$$

such that

$$c_j \in Y_j \setminus h_j(X_j)$$

$\forall j \in I$. Let

$$x \in \bigcup_{i \in I} X_i$$

such that $h(x) = c$.
Let j be such that $x \in X_j$. Then

$$h_j(x) = c_j \notin h_j(X_j),$$

a contradiction. $\blacksquare$