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Optimal Control

Rocket Car (1) Control: $u(t)$ = force, unknown
position: $w(t)$
time: t Final time $T>0$, given

$\ddot{w} = u$ (nondimensional form)

$$\begin{cases} w(0) = w_0 > 0 \\ \dot{w}(0) = 0 \end{cases} \qquad \begin{cases} w(T) = 0 \\ \dot{w}(T) = 0 \end{cases}$$

Minimize $\int_0^T u(t)^2\,dt$, fuel consumption

$\rightarrow$ minimize $\int_0^T \ddot{w}^2\,dt$ on

$$D = \{ w \in C^2[0,T];\ w(0) = w_0,\ \dot{w}(0) = \dot{w}(T) = w(T) = 0 \}$$

Integrand $f = \ddot{w}^2$

E-L equation: $$f_w - \frac{d}{dt} f_{\dot{w}} + \frac{d^2}{dt^2} f_{\ddot{w}} = 0$$

$$\rightarrow (2\ddot{w})^{\cdot\cdot} = 0 \rightarrow \ddddot{w} = 0$$

$$\rightarrow w = \frac{at^3}{6} + \frac{bt^2}{2} + ct + d$$

$$\begin{cases} w(0) = w_0 \\ \dot{w}(0) = 0 \end{cases} \rightarrow \begin{cases} d = w_0 \\ c = 0 \end{cases}$$

$$\begin{cases} w(T) = 0 \\ \dot{w}(T) = 0 \end{cases} \rightarrow \begin{cases} aT^3/6 + bT^2/2 + w_0 = 0 \\ aT^2/2 + bT = 0 \end{cases}$$

$$\rightarrow b = -aT/2,\quad a = 12w_0/T^3,\quad b = -6w_0/T^2$$

$$\therefore w = \frac{2w_0}{T^3}t^3 - \frac{3w_0}{T^2}t^2 + w_0$$

$$\& \quad u = \ddot{w} = \frac{12w_0}{T^3}t - \frac{6w_0}{T^2}$$

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Rocket Car (2)

Bounded Control: $|u(t)| \le 1 \ \forall t$
Minimize T, $T > 0$
$u(t)$, T are unknown

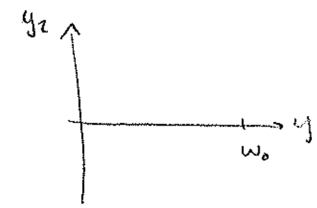
$\ddot{w} = u$ (nondimensional form)

$\begin{cases} w(0) = w_0 > 0 \\ \dot{w}(0) = 0 \end{cases}$ $\qquad \begin{cases} w(T) = 0 \\ \dot{w}(T) = 0 \end{cases}$

State: $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $y_1 = w$, $y_2 = \dot{w} \longrightarrow \dot{y}_2 = \ddot{w} = u$

$\dot{\vec{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$ 1st order system $\quad \vec{y}(0) = \begin{pmatrix} w_0 \\ 0 \end{pmatrix}$ $\vec{y}(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Try $\overset{*}{u}(t) = \begin{cases} -1 & \text{for } 0 \le t < t_s \\ 1 & \text{for } t_s < t \le T \end{cases}$



For $u = -1$:

$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -1 \end{cases}$ $\quad \dfrac{dy_1}{dy_2} = \dfrac{\dot{y}_1}{\dot{y}_2} = \dfrac{y_2}{-1} = -y_2$,

passing through $(w_0, 0)$

$\rightarrow y_1 = -\frac{1}{2} y_2^2 + w_0$

For $u = 1$: similarly $dy_1/dy_2 = y_2$, passes through $(0,0)$

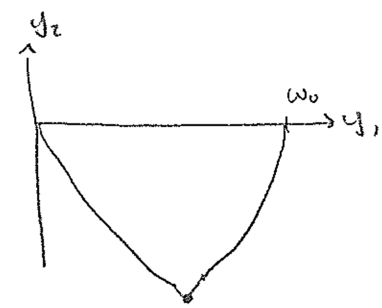
$\rightarrow y_1 = \frac{1}{2} y_2^2$

Intersection:

$$-\frac{y_2^2}{2} + w_0 = \frac{y_2^2}{2}$$

$$\rightarrow y_2^2 = w_0 \rightarrow y_2 = -\sqrt{w_0}$$

$$y_1 = \frac{w_0}{2}$$



In terms of t:

For $u = -1$, $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -1 \end{cases} \rightarrow \ddot{y}_1 = \dot{y}_2 = -1$

$\dot{y}_1 = -t + c_1 \rightarrow y_1 = -t^2/2 + c_1 t + c_2$

$y_2 = \dot{y}_1 = -t + c_1$

$\begin{cases} y_1(0) = w_0 \\ y_2(0) = 0 \end{cases} \longrightarrow \begin{cases} y_{1-} = -t^2/2 + w_0 \\ y_{2-} = -t \end{cases}$

Find t_s: $\begin{cases} -t_s^2/2 + w_0 = w_0/2 \\ -t_s = -\sqrt{w_0} \end{cases} \longrightarrow \begin{cases} t_s^2/2 = w_0/2 \longrightarrow t_s^2 = w_0 \\ t_s = \sqrt{w_0} \end{cases}$

For $u=1$: $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 1 \end{cases}$ $\quad \ddot{y}_1 = 1 \longrightarrow \begin{cases} y_{1+} = t^2/2 + d_1 t + d_2 \\ y_{2+} = t + d_1 \end{cases}$

At $t_s = \sqrt{w_0}$: $\begin{cases} y_{1+} = w_0/2 + d_1\sqrt{w_0} + d_2 = w_0/2 \\ y_{2+} = \sqrt{w_0} + d_1 = -\sqrt{w_0} \end{cases}$

$\longrightarrow d_1 = -2\sqrt{w_0}, \quad d_2 = -d_1\sqrt{w_0} = 2w_0$

Thus $\begin{cases} y_{1+} = t^2/2 - 2\sqrt{w_0}\,t + 2w_0 \\ y_{2+} = t - 2\sqrt{w_0} \end{cases}$

Find T: $\begin{cases} T^2/2 - 2\sqrt{w_0}\,T + 2w_0 = 0 \\ T - 2\sqrt{w_0} = 0 \end{cases}$

$\longrightarrow T = 2\sqrt{w_0}$

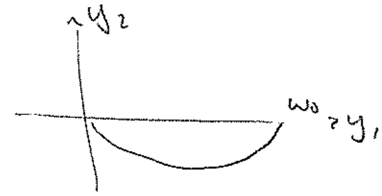
Let this solution be $\vec{y}^*$, $T^* = 2\sqrt{w_0}$

For a solution $\vec{y}$, T $\quad y_2(t) = \dot{y}_1(t) < 0$ for $0 < t < T$

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{u}{y_2} \longrightarrow y_2\frac{dy_2}{dy_1} = u \qquad |u| \le 1$$

$$\int_0^{y_1} y_2 \frac{dy_2}{dy_1}\,dy_1 \le \int_0^{y_1} dy_1 = y_1$$

$$\int_0^{y_2} y_2\,dy_2 = \frac{y_2}{2} \longrightarrow \boxed{y_1 \ge \frac{y_2}{2}}$$



Similarly, $$\int_{y_1}^{w_0} y_2\frac{dy_2}{dy_1}\,dy_1 \ge -\int_{y_1}^{w_0} dy_1 = -(w_0 - y_1)$$

$$\int_{y_2}^{0} y_2\,dy_2 = -\frac{y^2}{2} \longrightarrow y_1 - w_0 \le -\frac{y_2^2}{2}$$

$$\longrightarrow \boxed{y_1 \le w_0 - y_2^2/2}$$

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Now $$T = \int_0^T dt = \int_{w_0}^0 \frac{dy_1}{y_2} = \int_0^{w_0} \frac{dy_1}{(-y_2)} \ge \int_0^{w_0} \frac{dy_1}{(-y_2^*)} = \int_{w_0}^0 \frac{dy_1}{dy_2^*} = T^*$$

$\therefore T \ge T^*$

u^* is a bang-bang control

$T = \int_0^T dt.$

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General Formulation

Minimize $J(\vec{y},\vec{u}) = \int_0^T f(t,\vec{y},\vec{u})\,dt$, $\quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$

subject to $\quad \dot{\vec{y}} = \vec{g}(t,\vec{y},\vec{u})$

(a)
(b) $\left\{ \begin{array}{l} \text{\& either (a) } \vec{y}(0)=y_0,\ \vec{y}(T)=\vec{y}_T, \\ \quad \text{or (b) } \vec{y}(0)=y_0. \end{array} \right.$

Augmented Functional: $\quad \vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad \vec{p}=\vec{p}(t)$

$$\tilde{J}(\vec{y},\vec{u},\vec{p}) = \int_0^T \tilde{f}(t,\vec{y},\dot{\vec{y}},\vec{u},\vec{p})\,dt$$

$$\vec{f}(t,\vec{y},\dot{\vec{y}},\vec{u},\vec{p}) = f(t,\vec{y},\vec{u}) + \sum_{k=1}^{n} p_k\big(\dot{y}_k - g_k(t,\vec{y},\vec{u})\big)$$

$$\tilde{f} = f + \vec{p}^*(\dot{\vec{y}} - \vec{g}), \quad \vec{p}^* = (p_1 \cdots p_n)$$

Theorem: A stationary function for $\tilde{J}$ subject to (a) or (b) satisfies

(1) $(\tilde{f}_{\dot{y}_i})' = \tilde{f}_{y_i} \iff \dot{p}_i = f_{y_i} - \sum_{k=1}^{n} p_k (g_k)_{y_i} \qquad 1\le i\le n$

(2) $\tilde{f}_{u_i} = 0 \qquad \iff f_{u_i} - \sum_{k=1}^{n} p_k (g_k)_{u_i} = 0 \qquad 1\le i\le m$

(3) $\tilde{f}_{p_i} = 0 \quad \iff \dot{y}_i - g_i = 0 \iff \dot{y}_i = g_i \qquad 1\le i\le n$

Under case (b), $p_i(T)=0$, $1\le i\le n$

Proof: A stationary function for T under (a) or (b) satisfies

$$\frac{d}{d\alpha}\tilde{J}(\vec{y}+\alpha\vec{v}_1, \vec{u}+\alpha\vec{v}_2, \vec{p}+\alpha\vec{v}_3)\Big|_{\alpha=0} = 0$$

Under (a) $\vec{v}_1(0)=\vec{v}_1(T)=\vec{0}$

(b) $\vec{v}_1(0)=0$

$$\int_0^T \Big(\underline{\sum_{i=1}^{n} \tilde{f}_{y_i} v_i + \sum_{i=1}^{n} \tilde{f}_{\dot{y}_i} \dot{v}_i} + \sum_{i=1}^{m} \tilde{f}_{u_i} v_{2i} + \sum_{i=1}^{n} \tilde{f}_{p_i} v_{3i} \Big) dt = 0$$

Integrate by parts:

$$\int_0^T \sum_{i=1}^n \left(\tilde{f}_{y_i} - (\tilde{f}_{\dot{y}_i})^{\cdot}\right) v_{1i}\,dt + \int_0^T \sum_{i=1}^m \tilde{f}_{u_i} v_{2i}\,dt + \int_0^T \sum_{i=1}^n \tilde{f}_{p_i} v_{3i}\,dt$$
$$+ \sum_{i=1}^n f_{\dot{y}_i} v_{1i} \Big|_0^T = 0$$

In case (a), the last term is zero

Fundamental lemma $\Rightarrow \begin{cases} \tilde{f}_{y_i} - (\tilde{f}_{\dot{y}_i})^{\cdot} = 0 & 1 \le i \le n & \text{— (1)} \\ \tilde{f}_{u_i} = 0 & 1 \le i \le m & \text{— (2)} \\ \tilde{f}_{p_i} = 0 & 1 \le i \le n & \text{— (3)} \end{cases}$

In case (b), (1),(2),(3) hold and $v_{1i}(T) \neq 0 \rightarrow \tilde{f}_{\dot{y}_i}|_{t=T} = 0$
$\rightarrow p_i(T) = 0$
$1 \le i \le n$.

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Proposition: If $(\vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)$ minimizes $\tilde{J}$ then $(\vec{y}^\circ, \vec{u}^\circ)$ minimizes J subject to $\dot{\vec{y}} = \vec{g}(t, \vec{y}, \vec{u})$.

Proof: $(\vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)$ is stationary for $\tilde{J} \rightarrow \dot{\vec{y}}^\circ = \vec{g}(t, \vec{y}^\circ, \vec{u}^\circ)$

Consider $\vec{v}_1, \vec{v}_2$ st $(\vec{y}^\circ + \vec{v}_1)^{\cdot} = \vec{g}(t, \vec{y}^\circ + \vec{v}_1, \vec{u}^\circ + \vec{v}_2)$ & either

(a) $\vec{v}_1(0) = \vec{v}_1(T) = \vec{0}$ or (b) $\vec{v}_1(0) = \vec{0}$.

$$\begin{aligned} J(\vec{y}^\circ + \vec{v}_1, \vec{u}^\circ + \vec{v}_2) &= \tilde{J}(\vec{y}^\circ + \vec{v}_1, \vec{u}^\circ + \vec{v}_2, \vec{p}^\circ) \\ &\ge \tilde{J}(\vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ) \\ &= J(\vec{y}^\circ, \vec{u}^\circ). \end{aligned}$$

$\blacksquare$

Ex. Minimize $\int_0^T u^2\,dt$

subject to $\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$, $\vec{v}(0) = \begin{pmatrix} w_0 \\ 0 \end{pmatrix}$, $w_0 > 0$, $\vec{y}(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$f = u^2$, $\tilde{f} = u^2 + p_1(\dot{y}_1 - y_2) + p_2(\dot{y}_2 - u)$

E-L eqns for $\tilde{f}$: $(\tilde{f}_{\dot{y}_1})^{\cdot} = y_1 \longrightarrow \dot{p}_1 = 0$

$(\tilde{f}_{\dot{y}_2})^{\cdot} = y_2 \longrightarrow \dot{p}_2 = -p_1$

$\tilde{f}_u = 0 \longrightarrow 2u - p_2 = 0$

$\tilde{f}_{p_1} = 0 \longrightarrow \dot{y}_1 - y_2 = 0$

$\tilde{f}_{p_2} = 0 \longrightarrow \dot{y}_2 - u = 0$

Solve: $p_1 = c_1 \rightarrow \dot{p}_2 = -c_1 \rightarrow p_2 = -c_1 t + c_2$

$$u = \frac{p_2}{2} = \frac{-c_1 t + c_2}{2} = at + b$$

same as before

Ex. Minimize $\frac{1}{2}\int_0^T u^2 dt$

subject to $\dot{y} = -y + u$, $y(0) = 2$, $y(T) = 1$, $T > 0$ given

$f = \frac{1}{2}u^2$, $\tilde{f} = \frac{u^2}{2} + p(\dot{y} - (-y+u)) = \frac{u^2}{2} + p(\dot{y} + y - u)$

E-L eqn for $\tilde{f}$: $(\tilde{f}_{\dot{y}})' = \tilde{f}_y \rightarrow \dot{p} = p$

$\tilde{f}_u = 0 \rightarrow u - p = 0 \rightarrow u = p$

$\tilde{f}_p = 0 \rightarrow \dot{y} + y - u = 0 \rightarrow \dot{y} + y = u$

$p = ce^t \rightarrow u = ce^t$

$\dot{y} + y = ce^t$

$(e^t y)' = ce^{2t} \rightarrow e^t y = \frac{c}{2}e^{2t} + d \rightarrow y = \frac{c}{2}e^t + de^{-t}$

$y(0) = 2$: $\frac{c}{2} + d = 0$

$y(T) = 1$: $\frac{c}{2}e^T + de^{-T} = 1 \rightarrow \cdots \rightarrow c = \dfrac{2(1-2e^{-T})}{e^T - e^{-T}}$

$\rightarrow u = ce^t = \dfrac{2(1-2e^{-T})}{e^T - e^{-T}} e^t$

$y = \frac{c}{2}e^t + (2 - \frac{c}{2})e^{-t}$

$T = \ln 2 \rightarrow 1 - 2e^{-T} = 0 \rightarrow c = 0 \rightarrow u = 0$

Reformulation

$$\tilde{f}(t, \vec{y}, \dot{\vec{y}}, \vec{u}, \vec{p}) = f(t, \vec{y}, \vec{u}) + \sum_{k=1}^{n} p_k \left(\dot{y}_k - g_k(t, \vec{y}, \vec{u})\right)$$

Pontryagin Hamiltonian:

$$H = \sum_{i=1}^{n} \dot{y}_i \tilde{f}_{\dot{y}_i} - \tilde{f}$$

$$= \sum_{i=1}^{n} \dot{y}_i p_i - f - \sum_{k=1}^{n} p_k(\dot{y}_k - g_k)$$

$$= -f + \sum_{k=1}^{n} p_k g_k$$

$H = H(t, \vec{y}, \vec{u}, \vec{p})$

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Optimality Conditions:

$$\dot{p}_i = f_{y_i} - \sum_{k=1}^{n} p_k (g_k)_{y_i} = -H_{y_i} \qquad 1 \le i \le n$$

$$\dot{y}_i = g_i = H_{p_i} \qquad 1 \le i \le n$$

$$f_{u_i} - \sum_{k=1}^{n} p_k (g_k)_{u_i} = 0 \to H_{u_i} = 0 \qquad 1 \le i \le m$$

$$H = -f + \vec{p} * \vec{g}$$

$$\begin{cases} \dot{\vec{p}} = -H_{\vec{y}} \\ \dot{\vec{y}} = \vec{g} \\ H_{\vec{u}} = 0 \end{cases}$$

- If H does not depend explicitly on t then it is constant on trajectories

Ex. (cont.) $H = -\frac{u^2}{2} + p(y-u)$

$$\begin{cases} \dot{p} = -H_y \\ \dot{y} = H_p \\ H_u = 0 \end{cases} \longrightarrow \begin{cases} \dot{p} = p \\ \dot{y} = -y + u \\ -u + p = 0 \end{cases} \quad \text{as before}$$

$$\dot{H} = -u\underline{\dot{u}} + \dot{p}(-y+u) + p(-\dot{y} + \underline{\dot{u}}) = \dot{u}(\cancelto{0}{p-u}) + p(-y+u) - p(-y+u) = 0$$

$\to H = \text{const.}$

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Theorem: If $h = -H$ is convex wrt $\vec{y}, \vec{u}$ & $(\vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)$ satisfies the optimality conditions & BC's (a) or (b) then $(\vec{y}^\circ, \vec{u}^\circ)$ minimizes J subject to $\dot{\vec{y}} = \vec{g}$ & BC's (a) or (b).

Proof:
$$H = -f + \sum_{k=1}^{n} p_k g_k$$
$$h = f - \sum_{k=1}^{n} p_k g_k \rightarrow f = h + \sum_{k=1}^{n} p_k g_k$$

$$J(\vec{y}, \vec{u}) - J(\vec{y}^\circ, \vec{u}^\circ) = \int_0^T \left(f(t, \vec{y}, \vec{u}) - f(t, \vec{y}^\circ, \vec{u}^\circ)\right) dt$$
$$= \int_0^T \left(h(t, \vec{y}, \vec{u}, \vec{p}^\circ) - h(t, \vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)\right) dt + \int_0^T \sum_{k=1}^{n} p_k^\circ \left(g_k(t, \vec{y}, \vec{u}) - g_k(t, \vec{y}^\circ, \vec{u}^\circ)\right) dt$$
$$\geq \int_0^T \Big(\sum_{i=1}^{n} \underbrace{h_{y_i}(t, \vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)}_{\dot{p}_i^\circ} (y_i - y_i^\circ) + \sum_{i=1}^{m} \cancel{h_{u_i}(t, \vec{y}^\circ, \vec{u}^\circ, \vec{p}^\circ)}^{0} (u_i - u_i^\circ) \Big) dt$$
$$+ \int_0^T \sum_{k=1}^{n} p_k^\circ \underbrace{\left(g_k(t, \vec{y}, \vec{u}) - g_k(t, \vec{y}^\circ, \vec{u}^\circ)\right)}_{\dot{y}_k - \dot{y}_k^\circ} dt$$

$(\dot{p}_i = -H_{y_i} = h_{y_i}, \; H_{u_i} = 0 \rightarrow h_{u_i} = 0)$

$$= \int_0^T \left(\sum_{i=1}^{k} \dot{p}_i^\circ (y_i - y_i^\circ) + p_i^\circ (\dot{y}_i - \dot{y}_i^\circ) \right) dt$$
$$= \int_0^T \sum_{i=1}^{n} \frac{d}{dt} \left(p_i^\circ (y_i - y_i^\circ) \right) dt$$
$$= \sum_{i=1}^{n} p_i^\circ(T)(y_i(T) - y_i^\circ(T)) - \sum_{i=1}^{n} p_i^\circ(0)(y_i(0) - y_i^\circ(0))$$
$= 0$ in case a or case b ∎

Ex. (cont.)
$$H = -\frac{u^2}{2} + p(-y+u)$$
$$h = -H = \frac{u^2}{2} + p(y-u)$$
h is convex with respect to y, u
$$\nabla^2 h = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{pos. semi-def.}$$

→ solution to optimality conditions is a minimizer

Ex. Rocket Car $\swarrow \dot{y}_1?$

$$\tilde{f} = u^2 + p_1(\dot{y}_1 - y_2) + p_2(\dot{y}_2 - u)$$

$$f = u^2$$

$$H = -u^2 + p_1 y_2 + p_2 u$$

$$h = -H = u^2 - p_1 y_2 - p_2 u$$

$$\nabla^2_{(y_1,y_2,u)} h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{pos. semi-def.}$$

h is convex wrt (y_1, y_2, u)

By thm, the solution to the optimality conditions is a minimizer

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If T is variable, then $H|_{t=T} = 0$ [Transversality Condition]

↳ see A5

Ex. Minimize $\int_0^T dt = T$

subject to $\dot{y} = u$, $y(0) = 1$, $y(T) = 0$, $T > 0$

$H = -1 + pu$, $\quad H = \text{const.} \rightarrow H = 0$ (by transversality condition)

$$\begin{cases} \dot{p} = -H_y = 0 \\ \dot{y} = H_p = u \\ 0 = H_u = p \end{cases} \longrightarrow \quad -1 + 0 = 0 \quad \times$$

$\therefore$ No optimal solution