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$$\rightarrow \begin{cases} -y + \dfrac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_1 \\ x + \dfrac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -c_2 \end{cases}$$

$$\begin{cases} y + c_1 = \dfrac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ x + c_2 = \dfrac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \end{cases} \longrightarrow (x+c_2)^2 + (y+c_1)^2 = \lambda^2 \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2+\dot{y}^2} = \lambda^2$$

take $\lambda > 0$: $2\pi\lambda = L \longrightarrow \lambda = L/2\pi$ $\quad A = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$

Isoperimetric Inequality: A: area enclosed by the curve

$$A \le L^2/4\pi$$

$$J(\vec{y}) = \int_a^b f(t, \vec{y}, \dot{\vec{y}})\,dt$$

E-L eqn. $(f_{\dot{y}_i})' = f_{y_i} \quad 1 \le i \le n$

Proposition: If $f = f(\vec{y}, \dot{\vec{y}})$ then

$$H = \sum_{i=1}^n \dot{y}_i f_{\dot{y}_i} - f$$

is constant on trajectories.

Proof: $$\frac{d}{dt} H = \sum_{i=1}^n \left(\cancel{\ddot{y}_i f_{\dot{y}_i}} + \dot{y}_i (f_{\dot{y}_i})'\right) - \sum_{i=1}^n \left(f_{y_i}\dot{y}_i + \cancel{f_{\dot{y}_i}\ddot{y}_i}\right) = 0$$

= by E-L eqn

<u>Hamilton's Principle</u>

Motion of a point mass of mass m

Conservative force $\vec{F}$: $\vec{F} = -\nabla U$, U is a potential for $\vec{F}$,

U is the potential energy

$L = T - U$, T is kinetic energy, L is the Lagrangian

Action: $$A(\vec{y}) = \int_a^b L(t, \vec{y}, \dot{\vec{y}})\,dt$$

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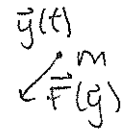
Hamilton's Principle: for all trajectories with $\vec{y}(a)$ & $\vec{y}(b)$ fixed the trajectory followed by the ~~path~~ point mass makes the action stationary, $\forall$ times a, b $a<b$



$$\vec{y}(t) = (y_1(t), y_2(t), y_3(t))$$

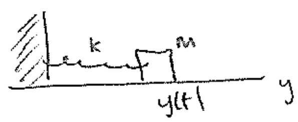
$$L = \tfrac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2) - U(y_1, y_2, y_3)$$

E-L eqn's $\quad (L_{\dot{y}_i})^{\cdot} = L_{y_i} \qquad 1 \le i \le 3$

$$\begin{cases} (m\dot{y}_1)^{\cdot} = -U_{y_1} \\ (m\dot{y}_2)^{\cdot} = -U_{y_2} \\ (m\dot{y}_3)^{\cdot} = -U_{y_3} \end{cases} \rightarrow \begin{cases} m\ddot{y}_1 = F_1 \\ m\ddot{y}_2 = F_2 \\ m\ddot{y}_3 = F_3 \end{cases} \rightarrow m\ddot{\vec{y}} = \vec{F}(\vec{y})$$

Newton's 2nd Law

Ex. Mass-Spring System



y is distance from equilibrium, $F = -ky$, $k>0$

$$F = -\frac{dU}{dy} = -ky \longrightarrow \frac{dU}{dy} = ky \longrightarrow U = ky^2/2$$

$$L = \tfrac{1}{2}m\dot{y}^2 - ky^2/2$$

E-L eqn $(L_{\dot{y}})^{\cdot} = L_y \rightarrow (m\dot{y})^{\cdot} = -ky \rightarrow \boxed{m\ddot{y} = -ky}$

doesn't depend on t explicitly

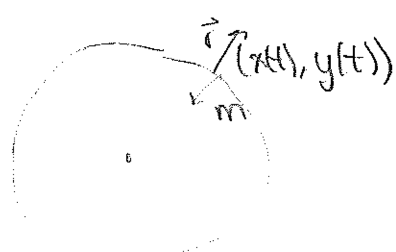
$$\longrightarrow H = \dot{y}L_{\dot{y}} - L = \dot{y}(m\dot{y}) - (\tfrac{1}{2}m\dot{y}^2 - ky^2/2) = \tfrac{1}{2}m\dot{y}^2 + ky^2/2 = c$$

on trajectory

Total energy is conserved $(H = T + U)$

<u>Kepler's Laws</u>

Motion of a planet about the sun



$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$\vec{F} = -k/r^2\,\hat{r}, \quad k>0$$

$$\vec{F} = -\nabla U, \quad U = -\frac{k}{r} \qquad r = \sqrt{x^2+y^2}$$

check: $\nabla U = U_r \hat{r} = \frac{k}{r^2}\hat{r}$

$$L = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + k/r$$

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$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad U = -k/r \qquad k = GMm > 0$

mass of sun

mass of planet

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$

E-L eqns $\begin{cases} \frac{d}{dt}(L_{\dot{\theta}}) = L_\theta \\ \frac{d}{dt}(L_{\dot{r}}) = L_r \end{cases} \longrightarrow \begin{cases} \frac{d}{dt}(mr^2\dot{\theta}) = 0 \rightarrow mr^2\dot{\theta} = p \\ m\ddot{r} = mr\dot{\theta}^2 - \frac{k}{r^2} \quad \text{(not used)} \end{cases}$

p is angular momentum, const.

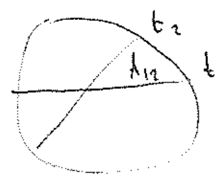
L doesn't depend explicitly on t

$\dot{r}L_{\dot{r}} + \dot{\theta}L_{\dot{\theta}} - L = c_1$

$\dot{r}(m\dot{r}) + \dot{\theta}(mr^2\dot{\theta}) - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}) - \frac{k}{r} = c_1$

$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} = c_1$ ← total energy

Suppose $r = r(\theta)$, $\int_{t_1}^{t_2} mr^2\frac{d\theta}{dt}dt = \int_{t_1}^{t_2} p\,dt = p(t_2 - t_1)$



$$\int_{\theta_1}^{\theta_2} mr^2\,d\theta$$

$$A_{12} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta \Rightarrow A_{12} = \frac{2p}{m}(t_2 - t_1)$$

Kepler's 2nd Law. equal areas are swept out in equal times

$\dot{r}^2 = \frac{2}{m}\left(c_1 - \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r}\right)$

$\frac{m}{2}r^2\dot{\theta}^2 = \frac{p}{2}\dot{\theta} = \frac{p}{2}\left(\frac{p}{mr^2}\right) = \frac{p^2}{2mr^2}$

$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} =$

$\frac{dr}{d\theta} = \frac{mr^2}{p}\sqrt{\frac{2}{m}\left(c_1 - \frac{p^2}{2mr^2} + \frac{k}{r}\right)}$, a separable ODE

Let $z = \frac{1}{r} \rightarrow \frac{dz}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$\frac{dz}{d\theta} = -\frac{\sqrt{2m}}{p}\sqrt{c_1 - \frac{p^2}{2m}z^2 + kz}$

$$\frac{-dz}{\sqrt{\frac{2mc_1}{p^2} - \left(z^2 - \frac{2km}{p^2}z\right)}} = dt$$

$$\int \frac{-dz}{\sqrt{\frac{2mc_1}{p^2}+\left(\frac{km}{p^2}\right)^2-\left(z-\frac{km}{p^2}\right)^2}} = \int d\theta$$

$$c_2 = \frac{2mc_1}{p^2}+\left(\frac{km}{p^2}\right)^2 \qquad c_2>0$$

let $z-\frac{km}{p^2} = \sqrt{c_2}\cos u$

$$dz = -\sqrt{c_2}\sin u$$

$$\int \frac{\sqrt{c_2}\sin u}{\sqrt{c_2}\sin u}\,du = \int d\theta \longrightarrow u = \theta-\theta_0$$

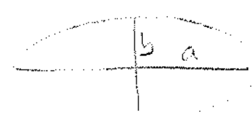
$$\cos^{-1}\left(\frac{z-km/p^2}{\sqrt{c_2}}\right) = \theta-\theta_0$$

$$z-\frac{km}{p^2} = \sqrt{c_2}\cos(\theta-\theta_0) \to r = \frac{1}{\frac{km}{p^2}+\sqrt{c_2}\cos(\theta-\theta_0)} = \frac{c}{1+\varepsilon\cos(\theta-\theta_0)}$$

$$c = \frac{p^2}{km}, \quad \varepsilon = \frac{p^2}{km}\sqrt{\frac{2mc_1}{p^2}+\left(\frac{km}{p^2}\right)^2} = \sqrt{1+\frac{2p^2}{mk^2}c_1}$$

Take $\theta_0=0$, $c>0$ $\boxed{r = \frac{c}{1+\varepsilon\cos\theta}}$

$\varepsilon=0$: circle
$0<\varepsilon<1$: ellipse
$\varepsilon=1$: paraboda
$\varepsilon>1$: hyperbola



a = semi-major axis length
b = semi-minor axis length

K's 1st Law: orbits of planets are ellipses

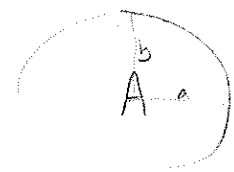
Consider $0<\varepsilon<1$: $r(1+\varepsilon\cos\theta) = c$

$$\varepsilon r\cos\theta - c = -r$$
$$(\varepsilon x - c)^2 = r^2 = x^2+y^2$$
$$\varepsilon x^2 - 2\varepsilon c x + c^2 = x^2+y^2$$
$$(1-\varepsilon^2)x^2 + 2\varepsilon c x + y^2 = c^2$$
$$(1-\varepsilon^2)\left(x^2+\frac{2\varepsilon c}{(1-\varepsilon^2)}\right)+y^2 = (1-\varepsilon)^2\left(x-\frac{\varepsilon c}{1-\varepsilon^2}\right)^2+y^2 = c^2$$

?

$$\left(x-\frac{\varepsilon c}{1-\varepsilon^2}\right)^2+\frac{y^2}{1+\varepsilon^2} = \frac{\varepsilon^2c^2}{(1-\varepsilon^2)^2}+\frac{c^2}{1+\varepsilon^2}$$

$$x = \frac{-\varepsilon c}{1-\varepsilon^2} \longrightarrow y = b = \frac{c}{\sqrt{1-\varepsilon^2}}$$



$$a^2 = \frac{c^2}{(1-\varepsilon^2)^2} \longrightarrow a = \frac{c}{1-\varepsilon^2}$$

T = period

K's 2nd Law: $A = \frac{p}{2m} T$

$$A = \pi a b = \frac{\pi c^2}{(1-\varepsilon)^{3/2}} = \frac{\pi c^2}{\left(\frac{c}{a}\right)^{3/2}} = \pi c^{1/2} a^{3/2}$$

$$A^2 = \pi^2 c a^3 = \frac{p^2}{4m^2} T^2$$

$$a^3 = \frac{p^2}{4\pi^2 m^2} T^2 \underset{c = p^2/4m}{=} \frac{1}{4\pi^2} \frac{k}{m} T^2 \underset{k = GMm}{=} \frac{GM}{4\pi^2} T^2$$

K's 3rd Law: $a^3 \propto T^2$ for all planets

"I don't think I've ever seen a formal proof of this, but it always seems to work."

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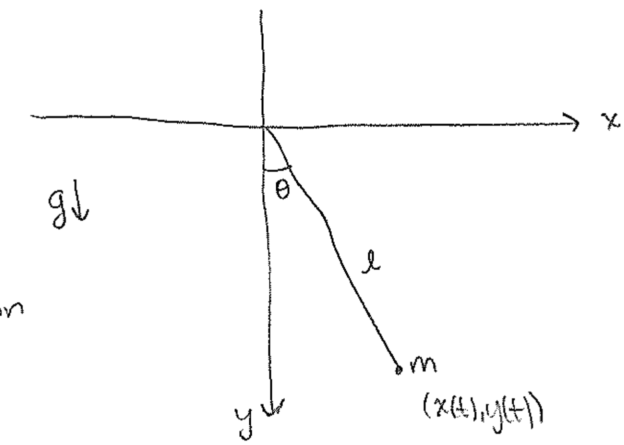
Generalized Coordinates

Ideal Pendulum



$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$U = -mgy + \text{const.}$

$= mg(-y+\ell)$

$U = 0$ for $y=\ell$, down position

Constraint: $x^2+y^2=\ell^2$

Use θ instead of (x,y)

$\begin{cases} x = \ell\sin\theta \\ y = \ell\cos\theta \end{cases} \longrightarrow \begin{cases} \dot{x} = \ell\dot{\theta}\cos\theta \\ \dot{y} = \ell\dot{\theta}\sin\theta \end{cases}$

$\rightarrow T = \frac{1}{2}m\ell^2\dot{\theta}^2, \quad U = mg\ell(1-\cos\theta)$

$\rightarrow L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$

E-L eqn: $\frac{d}{dt}(L_{\dot{\theta}}) = L_\theta$

$$\Rightarrow \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) = -mg\ell\sin\theta \longrightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

Linearization about $\theta=0$: $\sin\theta \approx \theta$

$\ddot{\theta} + \frac{g}{\ell}\theta = 0$

$\omega := \sqrt{g/\ell} \longrightarrow \ddot{\theta} + \omega^2\theta = 0$

$\theta = c_1\cos\omega t + c_2\sin\omega t$

periodic with period $2\pi/\omega$

Generalized coordinates $\vec{q} = (q_1, \ldots, q_n)$

(i) q_i's are independent

(ii) n is the smallest number of coordinates

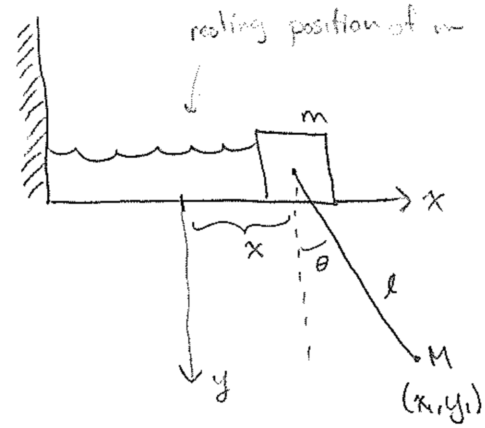
Hamilton's Principle Revised:

Between any two fixed times $a<b$, a system moves along trajectories which make the

$$A(\vec{q}) = \int_a^b L\,dt, \quad L = T - U$$

stationary

Spring-Mass-Pendulum System



Generalized Coords: x, θ

spring force: $-h(x)$

$$U_{spring}(x) = -\int_0^x -h(s)ds = \int_0^x h(s)ds$$

Linear spring: $h(x) = kx, \quad k>0$

$$U = \int_0^x h(s)ds + Mg\ell(1-\cos\theta)$$

$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}M(\dot{x}_1^2 + \dot{y}_1^2)$$

$$\begin{cases} x_1 = x + \ell\sin\theta \\ y_1 = \ell\cos\theta \end{cases} \longrightarrow \begin{cases} \dot{x}_1 = \dot{x} + \ell\dot{\theta}\cos\theta \\ \dot{y}_1 = -\ell\dot{\theta}\sin\theta \end{cases}$$

$$\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2 + \dot{x}2\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\cos^2\theta + \ell^2\dot{\theta}^2\sin^2\theta$$
$$= \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

$$\longrightarrow L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + 2\ell\dot{\theta}\dot{x}\cos\theta + \ell^2\dot{\theta}^2)$$
$$- Mg\ell(1-\cos\theta) - \int_0^x h(s)ds$$

E-L eqns: $(L_{\dot{x}})' = L_x \longrightarrow (m\dot{x} + M\dot{x} + M\ell\dot{\theta}\cos\theta)' = -h(x)$

$$\longrightarrow (m+M)\ddot{x} + M\ell\ddot{\theta}\cos\theta - M\ell\dot{\theta}^2\sin\theta = -h(x)$$

$(L_{\dot{\theta}})' = L_\theta \longrightarrow (M\ell\dot{x}\cos\theta + M\ell^2\dot{\theta})' = -Mg\ell\sin\theta - M\ell\dot{\theta}\dot{x}\sin\theta$

$$\longrightarrow M\ell\ddot{x}\cos\theta - \cancel{M\ell\dot{x}\dot{\theta}\sin\theta} + M\ell^2\ddot{\theta} = -Mg\ell\sin\theta - \cancel{M\ell\dot{\theta}\dot{x}\sin\theta}$$

$$\longrightarrow \begin{cases} (m+M)\ddot{x} + M\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = -h(x) \\ (\cos\theta)\ddot{x} + \ell\ddot{\theta} = -g\sin\theta \end{cases}$$

Assume Linear spring: $h(x) = kx$

Linearization about $(x,\theta) = (0,0)$ $\qquad \theta, \dot{\theta}, \ddot{\theta} = \mathcal{O}(\varepsilon)$

$\sin\theta \approx \theta \qquad \cos\theta \approx 1 \qquad \dot{\theta}^2\theta = \mathcal{O}(\varepsilon^3)$

$$\begin{cases} (m+M)\ddot{x} + M\ell\ddot{\theta} = -kx \\ \ddot{x} + \ell\ddot{\theta} = -g\theta \end{cases}$$

$$\begin{pmatrix} m+M & M\ell \\ 1 & \ell \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -kx \\ -g\theta \end{pmatrix}$$

$$\begin{pmatrix} m+M & M\ell \\ 1 & \ell \end{pmatrix}^{-1} = \frac{1}{m\ell}\begin{pmatrix} \ell & -M\ell \\ -1 & m+M \end{pmatrix}$$

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$$\rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \frac{1}{m\ell}\begin{pmatrix} \ell & -M\ell \\ -1 & m+M \end{pmatrix}\begin{pmatrix} -kx \\ -g\theta \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & \frac{Mg}{m} \\ \frac{k}{m\ell} & -\frac{(m+M)g}{m\ell} \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x \\ \theta \end{pmatrix}, \quad \ddot{\vec{x}} = A\vec{x}, \quad A = \begin{pmatrix} \nearrow \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Try $\vec{x} = (\cos\omega t)\vec{v} \rightarrow -\omega^2 \cancel{\cos\omega t}\, \vec{v} = \cancel{\cos\omega t}\, A\vec{v}$

$\rightarrow A\vec{v} = -\omega^2\vec{v} \rightarrow \omega^2 = -\lambda$, λ an eigenvalue of A with eigenvector $\vec{v}$

$\det(A-\lambda I) = 0$

$\rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$

$\rightarrow \lambda^2 - (a_{11}+a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0$

$$\rightarrow \lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2}$$

$$(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21}) = (a_{11}-a_{22})^2 + 4a_{12}a_{21}$$

$a_{12}, a_{21} > 0 \rightarrow \lambda$'s are real.

$$a_{11}a_{22} - a_{12}a_{21} = \frac{k}{m^2\ell}(m+M)g - \frac{kMg}{m^2\ell} = \frac{kgm}{m^2\ell} > 0$$

$a_{11}, a_{22} < 0 \rightarrow \lambda_1 < \lambda_2 < 0$

$\omega_1^2 = -\lambda_1$, $\omega_2^2 = -\lambda_2 \rightarrow \omega_1 = \sqrt{-\lambda_1}$, $\omega_2 = \sqrt{-\lambda_2} \rightarrow \omega_1 > \omega_2 > 0$

eigenvectors $\vec{v}_1, \vec{v}_2$ correspond to λ_1, λ_2

General solutions: $\vec{x}(t) = (c_1\cos\omega_1 t + c_2\sin\omega_1 t)\vec{v}_1 + (c_3\cos\omega_2 t + c_4\sin\omega_2 t)\vec{v}_2$

Two modes: $(c_1\cos\omega_1 t + c_2\sin\omega_1 t)\vec{v}_1$

$(c_3\cos\omega_2 t + c_4\sin\omega_2 t)\vec{v}_2$

These are each periodic, frequencies ω_1, ω_2 respectively.

The general motion is periodic if $n\omega_1 = m\omega_2$, positive integers n, m, otherwise the motion is quasi-periodic.

Canonical Equations

One generalized coordinate q

Lagrangian: $L(t,q,\dot q) = \frac{1}{2}a(t,q)\dot q^2 - U(t,q)$, $a(t,q)>0$

Define $p = L_{\dot q} = a(t,q)\dot q \longrightarrow \dot q = \frac{p}{a(t,q)}$ generalized momentum

E-L eqn: $\frac{d}{dt}\left(L_{\dot q}\right) = L_q \longrightarrow \dot p = L_q$

Thus $\dot q = \frac{p}{a(t,q)}$, $\dot p = L_q\left(t,q,\frac{p}{a(t,q)}\right)$

a 1st order system for (q,p)

Define $H = \dot q L_{\dot q} - L = a(t,q)\dot q^2 - \left(\frac{1}{2}a(t,q)\dot q^2 - U(t,q)\right)$
$= \frac{1}{2}a(t,q)\dot q^2 + U(t,q)$

$\rightarrow H = \frac{p^2}{2a(t,q)} + U$ H is the Hamiltonian

$\nwarrow$ H as a function of t, q, p

Note: $\frac{\partial H}{\partial p} = \frac{p}{a(t,q)}$, $\frac{\partial H}{\partial q} = -\frac{p^2}{2a(t,q)^2}a_q(t,q) + U_q(t,q)$

$L_q = \frac{1}{2}a_q(t,q)\dot q^2 - U_q(t,q)$

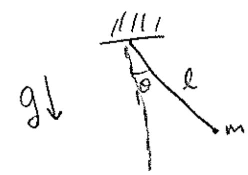
$= \frac{a_q(t,q)}{2a(t,q)^2}p^2 - U_q(t,q) = -\frac{\partial H}{\partial q}$

Thus

$$\begin{cases} \dot q = \partial H/\partial p \\ \dot p = -\partial H/\partial q \end{cases}$$

Canonical Equations
1st order system for (q,p)

Ex Ideal pendulum



$T = \frac{1}{2}m\ell^2\dot\theta^2$

$U = mg\ell(1-\cos\theta)$

$L = T - U = \frac{1}{2}m\ell^2\dot\theta^2 - mg\ell(1-\cos\theta)$

$p = L_{\dot\theta} = m\ell^2\dot\theta$ $\quad\dot\theta = p/m\ell^2$

$$H = \dot{\theta} L_{\dot\theta} - L = m\ell^2\dot\theta^2 - \tfrac{1}{2} m\ell^2\dot\theta^2 + mg\ell(1-\cos\theta)$$
$$= \tfrac{1}{2} m\ell^2\dot\theta^2 + mg\ell(1-\cos\theta)$$
$$H = \frac{1}{2}\frac{p^2}{m\ell^2} + mg\ell(1-\cos\theta).$$

canonical equations: $\dot\theta = H_p = \frac{p}{m\ell^2}$

$\dot p = -H_\theta = -mg\ell\sin\theta$

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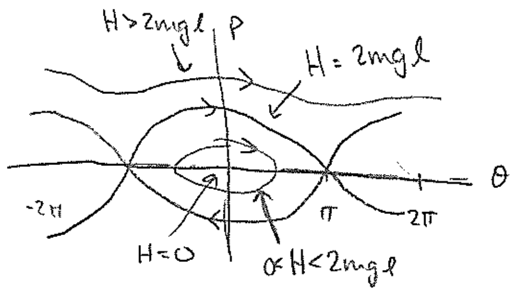
check: $\ddot\theta = \frac{\dot p}{m\ell^2} = \frac{-mg\ell\sin\theta}{m\ell^2} = -\frac{g}{\ell}\sin\theta$ ✓

equilibrium points: $\dot\theta = 0, \dot p = 0$

$\rightarrow p = 0, \quad \sin\theta = 0 \rightarrow \theta = n\pi, \; n \in \mathbb{Z}$

No explicit dependence on t in $L \rightarrow$ H = constant on trajectories

H = total energy



$$\begin{cases} \dot q = \frac{\partial H}{\partial p} \\ \dot p = -\frac{\partial H}{\partial q} \end{cases} \qquad \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \vec f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\operatorname{div} \vec f = \frac{\partial}{\partial q}(f_1) + \frac{\partial}{\partial p}(f_2) = H_{pq} - H_{qp} = 0$$

If H doesn't depend on t explicitly then the flow preserves areas.

<u>n Generalized Coordinates</u> $\vec q = (q_1, \dots, q_n)$

Lagrangian: $L(t, \vec q, \dot{\vec q}) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}\dot q_i \dot q_j - U(t, \vec q)$

Assume $A = (a_{ij})$ is symmetric and positive definite

Define $p_k = L_{\dot q_k} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}(\delta_{ik}\dot q_j + \delta_{jk}\dot q_i)$

$$= \frac{1}{2}\left(\sum_{j=1}^{n} a_{kj}\dot{q}_j + \sum_{i=1}^{n} \overset{\overset{a_{ki}}{\shortparallel}}{a_{ik}}\dot{q}_i\right)$$

$$= \sum_j a_{kj}\dot{q}_j$$

So $\vec{p} = A\dot{\vec{q}} \longrightarrow \dot{\vec{q}} = A^{-1}\vec{p}_{\!} = \vec{G}(t,\vec{q},\vec{p})$

Let $\vec{G} = (g_1, \dots, g_n)$

$$\dot{q}_i = g_i(t,\vec{q},\vec{p}) \qquad 1 \le i \le n \qquad \text{——— (1)}$$

E-L eqns: $\dfrac{d}{dt}\left(L_{\dot{q}_i}\right) = L_{q_i}$

$$\longrightarrow \quad \dot{p}_i = L_{q_i}\left(t,\vec{q},\vec{G}(t,\vec{q},\vec{p})\right) \qquad 1 \le i \le n \qquad \text{——— (2)}$$

(1) & (2) are a first order system for $(\vec{q},\vec{p})$.

Define

$$H = \sum_{j=1}^{n} \dot{q}_j \underbrace{L_{\dot{q}_j}}_{=p_j} - L = \sum_{j=1}^{n} p_j g_j - L(t,\vec{q},\vec{G})$$

Note:

$$\frac{\partial H}{\partial p_i} = g_i + \sum_{j=1}^{n} p_j \frac{\partial g_j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\frac{\partial g_j}{\partial p_i} = g_i$$

$$\frac{\partial H}{\partial q_i} = \sum_{j=1}^{n} p_j \frac{\partial g_j}{\partial q_i} - \left(L_{q_i} + \sum_{j=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j}\frac{\partial g_j}{\partial q_i}\right) = -L_{q_i}$$

Thus

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} & 1 \le i \le n \\[2ex] \dot{p}_i = -\dfrac{\partial H}{\partial q_i} & 1 \le i \le n \end{cases} \qquad \text{——(3)}$$

2 Canonical Equations

(3) is equivalent to (1) & (2). 1^{st} order system for $(\vec{q},\vec{p})$

If H doesn't depend ~~on~~ explicitly on t, H = constant on trajectories & the flow preserves volumes. (Liouville's Thm)

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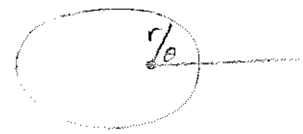
Ex. Motion of a planet about the sun

Generalized coordinates: r, θ

$$L = \tfrac{1}{2} m \left(\dot r^2 + r^2 \dot\theta^2\right) + \frac{k}{r}, \quad k>0$$



$$p_r = L_{\dot r}, \quad p_\theta = L_{\dot\theta}$$

$$\longrightarrow p_r = m\dot r, \quad p_\theta = m r^2 \dot\theta$$

$$\longrightarrow \dot r = p_r/m, \quad \dot\theta = p_\theta / m r^2$$

$$H = \dot r L_{\dot r} + \dot\theta L_{\dot\theta} - L = \frac{p_r}{m} p_r + \frac{p_\theta}{m r^2} p_\theta - \left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{k}{r}\right)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

Note: $H = T + U$. No explicit t in $H \to H = \text{const.}$ on any trajectory

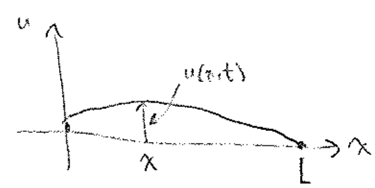
$\to$ Canonical equations of motion:

$$\begin{cases} \dot r = H_{p_r} = p_r/m \\ \dot\theta = H_{p_\theta} = p_\theta / m r^2 \\ \dot p_r = -H_r = -\left(-p_\theta^2/mr^3 + k/r^2\right) = p_\theta^2/mr^3 - k/r^2 \\ \dot p_\theta = -H_\theta = 0 \end{cases}$$

$p_\theta = c$, p_θ is angular momentum

Spatially Distributed Systems

Vibrating String



$u(x,t)$ is vertical displacement of the strings

$0 \le x \le L$

assume $|u|$ is small

ρ = density, mass per unit length

τ = tension

$$T = \int_0^L \tfrac{1}{2} u_t^2 \rho \, dx = \int_0^L \frac{\rho}{2} u_t^2 \, dx$$

$$U = \tau\left(\int_0^L \sqrt{1+u_x^2}\, dx - L\right) = \tau \int_0^L \Big(\underbrace{\sqrt{1+u_x^2} - 1}_{\approx 1 + \frac{u_x^2}{2} - 1 = \frac{u_x^2}{2}}\Big) dx \approx \int_0^L \frac{\tau}{2} u_x^2 \, dx$$

$$L = T - U = \int_0^L \left(\frac{\rho}{2}u_t^2 - \frac{\tau}{2}u_x^2\right)dx$$

For times $a<b$,

$$A(u) = \int_a^b L\,dt = \int_a^b\left(\int_0^L\left(\frac{\rho}{2}u_t^2 - \frac{\tau}{2}u_x^2\right)dx\right)dt$$

Hamilton's Principle: For times $a<b$, given $u(x,b)$, $u(x,a)$, $0 \le x \le L$ the system evolves such that $A(u)$ is stationary.

$$\delta A(u;v) = \frac{d}{d\alpha}A(u+\alpha v)\Big|_{\alpha=0}$$

$$A(u+\alpha v) = \int_a^b\int_0^L\left(\frac{\rho}{2}(u_t+\alpha v_t)^2 - \frac{\tau}{2}(u_x+\alpha v_x)^2\right)dx\,dt$$

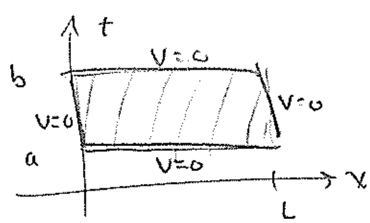
$$\frac{d}{d\alpha}A(u+\alpha v) = \int_a^b\int_0^L\left(\rho(u_t+\alpha v_t)v_t - \tau(u_x+\alpha v_x)v_x\right)dx\,dt$$

$$\Rightarrow \delta A(u;v) = \int_a^b\int_0^L\left(\rho u_t v_t - \tau u_x v_x\right)dx\,dt$$

(i) B.C. $u(0,t) = u(L,t) = 0 \quad \forall t$

$\delta A(u;v) = 0 \quad \forall v$ st $v(0,t) = v(L,t) = 0 \quad \forall t$

Also $v(x,b) = v(x,a) = 0 \quad \forall x \in [0,L]$



Integrate by parts

$$\int_a^b \rho u_t v_t\,dt = \overset{0}{\cancel{\rho u_t v\Big|_{t=a}^{t=b}}} - \int_a^b \rho u_{tt} v\,dt$$

$$\int_a^b\int_0^L \rho u_t v_t\,dx\,dt = -\int_a^b\int_0^L \rho u_{tt} v\,dx\,dt$$

$$\int_0^L -\tau u_x v_x\,dx = -\overset{0}{\cancel{\tau u_x v\Big|_0^L}} + \int_0^L \tau u_{xx} v\,dx$$

$$\int_a^b\int_0^L(-\tau u_x v_x)\,dx\,dt = \int_a^b\int_0^L \tau u_{xx} v\,dx\,dt$$

$$0 = \delta A(u;v) = \int_a^b\int_0^L\left(-\rho u_{tt} + \tau u_{xx}\right)v\,dx\,dt$$

Variant of Fund. Lemma $\Longrightarrow -\rho u_{tt} + \tau u_{xx} = 0$

$$\boxed{u_{tt} = \frac{\tau}{\rho}u_{xx}} \qquad \begin{array}{l} 0 \le x \le L \\ a \le t \le b \end{array}$$

a,b arbitrary $\Rightarrow u_{tt} = \tau/\rho\, u_{xx} \quad 0 \le x \le L \quad \forall t$

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(2) B.C. $u(0,t)=0$, no condition at $x=L$
(free end condition)

$$\Rightarrow \delta A(u;v) = \int_a^b \int_0^L (\rho u_t v_t - \tau u_x v_x)\,dx\,dt$$

First take v s.t. $v(L,t)=0 \Rightarrow u_{tt} = \tau/\rho\, u_{xx}$, $0 \le x \le L$, $\forall t$
(same as (1))

$$\delta A(u;v) = \int_a^b \int_0^L \cancelto{0}{(\rho u_{tt} - \tau u_{xx})}\, v\,dx\,dt + \int_a^b -\tau u_x(L,t)\,v(L,t)\,dt$$

$\Rightarrow u_x(L,t) = 0$ by Fundamental Lemma
$a \le t \le b$

a,b arbitrary $\Rightarrow u_x(L,t) = 0 \ \forall t$

[Natural Boundary Condition]