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Vector-Value Functions

$$\vec{y}(t) = (y_1(t), \dots, y_n(t)), \quad \text{dot} = \frac{d}{dt}$$
$$\dot{\vec{y}}(t) = (\dot{y}_1(t), \dots, \dot{y}_n(t))$$
$$J(\vec{y}) = \int_a^b f(t, \vec{y}, \dot{\vec{y}})\,dt = \int_a^b f(t, y_1(t), \dots, y_n(t), \dot{y}_1(t), \dots, \dot{y}_n(t))\,dt$$
$$\vec{y} \in (C^1[a,b])^n, \quad y_i \in C^1[a,b], \quad 1 \le i \le n$$
$$f \in C^1([a,b] \times \mathbb{R}^{2n})$$

For $\vec{y}, \vec{v} \in (C^1[a,b])^n$,

$$\delta J(\vec{y}; \vec{v}) = \frac{d}{d\alpha} J[\vec{y} + \alpha\vec{v})\Big|_{\alpha=0}$$
$$J(\vec{y} + \alpha\vec{v}) = \int_a^b f(t, y_1 + \alpha v_1, \dots, y_n + \alpha v_n, \dot{y}_1 + \alpha\dot{v}_1, \dots, \dot{y}_n + \alpha\dot{v}_n)\,dt$$
$$\frac{d}{d\alpha} J(\vec{y} + \alpha\vec{v}) = \int_a^b \left(f_{y_1} v_1 + \dots + f_{y_n} v_n + f_{\dot{y}_1}\dot{v}_1 + \dots + f_{\dot{y}_n}\dot{v}_n\right)dt$$
$$\delta J(\vec{y}; \vec{v}) = \int_a^b \left(f_{y_1}[\vec{y}]v_1 + \dots + f_{y_n}[\vec{y}]v_n + f_{\dot{y}_1}[\vec{y}]\dot{v}_1 + \dots + f_{\dot{y}_n}[\vec{y}]\dot{v}_n\right)dt$$
$$[\vec{y}] = (t, \vec{y}, \dot{\vec{y}})$$

Suppose $\vec{y}_0$ is an <u>extremal</u> of $J(\vec{y})$ on $D = \{\vec{y} \in (C^1[a,b])^n ;\ \vec{y}(a) = \vec{y}_a, \vec{y}(b) = \vec{y}_b\}$

i.e. $\delta J(\vec{y}_0; \vec{v}) = 0 \quad \forall \vec{v} \in (C^1[a,b])^n$ st $\vec{y} + \vec{v} \in D$

equivalently $\vec{v}(a) = \vec{v}(b) = 0$

Integrate by parts:

$$0 = \delta J(\vec{y}_0, \vec{v}) = \int_a^b \left(\left(f_{y_1}[\vec{y}] - \frac{d}{dt}(f_{\dot{y}_1}[\vec{y}])\right)v_1 + \dots + \left(f_{y_n}[\vec{y}] - \frac{d}{dt}(f_{\dot{y}_n}[\vec{y}])\right)v_n\right)dt$$
$$+ \left(f_{\dot{y}_1}[\vec{y}]v_1\right)\Big|_a^b{}^{=0} + \dots + \left(f_{\dot{y}_n}[\vec{y}]v_n\right)\Big|_a^b{}^{=0}$$

Put $v_2 = \dots = v_n = 0$: $\int_a^b \left(f_{y_1}[\vec{y}] - \frac{d}{dt} f_{\dot{y}_1}[\vec{y}]\right)v_1\,dt$

Fundamental lemma $\Rightarrow$

$$f_{y_1}[\vec{y}] - \frac{d}{dt} f_{\dot{y}_1}[\vec{y}] = 0$$

In the same way,

$$\left.\begin{aligned} f_{y_1}[\vec{y}] - \frac{d}{dt} f_{\dot{y}_1}[\vec{y}] &= 0 \\ f_{y_2}[\vec{y}] - \frac{d}{dt} f_{\dot{y}_2}[\vec{y}] &= 0 \\ &\vdots \\ f_{y_n}[\vec{y}] - \frac{d}{dt} f_{\dot{y}_n}[\vec{y}] &= 0 \end{aligned}\right\}$$

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Ex. Find the shortest ~~~~ between two points (x_0, y_0) & (x_1, y_1).

path? curve?

curve $(x(t), y(t))$, $t \in [0,1]$

$$J(x,y) = \int_0^1 \sqrt{x'(t) + y'(t)}\, dt$$

Minimize J on $D = \{(x,y) \in (C^1[0,1])^2 ;\ (x(0), y(0)) = (x_0, y_0),\ (x(1), y(1)) = (x_1, y_1),\ \dot{x}^2 + \dot{y}^2 \neq 0\}$

$f = \sqrt{\dot{x}^2 + \dot{y}^2}$.

E-L eqns:
$$\begin{cases} f_x - \frac{d}{dt} f_{\dot{x}} = 0 \\ f_y - \frac{d}{dt} f_{\dot{y}} = 0 \end{cases} \longrightarrow \begin{matrix} \frac{d}{dt} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0 \\ \frac{d}{dt} \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0 \end{matrix}$$

$$\longrightarrow \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1, \quad \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2 \longrightarrow \text{If } C_1 = C_2 = 0 \text{ then } \dot{x}^2 + \dot{y}^2 = 0 \ \times$$

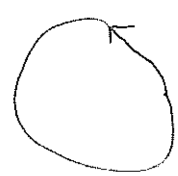
suppose $C_1 \neq 0$: $\frac{\dot{y}}{\dot{x}} = \frac{C_2}{C_1} = C \rightarrow \frac{dy}{dx} = C \rightarrow y = Cx + d$

$$y = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0) + y_0 \quad (\text{for } x_1 \neq x_0)$$

Ex Find the closed curve of given length that encloses the maximum area

Maximize $J(x,y) = \frac{1}{2} \int_0^1 (x\dot{y} - y\dot{x})\, dt$

on $D = \left\{ \begin{array}{l} (x,y) \in (C^1[0,1])^2 ;\ x(0) = x(1),\ y(0) = y(1), \\ \dot{x}^2 + \dot{y}^2 \neq 0,\ (x(t), y(t)) \text{ is a} \\ \text{simple closed curve} \end{array} \right\}$



$(x(t), y(t))$

subject to $\int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dx = L$

$f = \frac{1}{2}(x\dot{y} - y\dot{x}) \qquad g = \sqrt{\dot{x}^2 + \dot{y}^2}$

$\longrightarrow \tilde{f} = f + \lambda g = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda \sqrt{\dot{x}^2 + \dot{y}^2}$

E-L eqns: $\tilde{f}_x = \frac{d}{dt} \tilde{f}_{\dot{x}}$, $\tilde{f}_y = \frac{d}{dt} \tilde{f}_{\dot{y}}$

$$\frac{\dot{y}}{2} = \frac{d}{dt}\left(\frac{-y}{2} + \lambda \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \qquad -\frac{\dot{x}}{2} = \frac{d}{dt}\left(\frac{x}{2} + \lambda \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)$$

$$\rightarrow \begin{cases} -y + \dfrac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c_1 \\ x + \dfrac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -c_2 \end{cases}$$

$$\begin{cases} y + c_1 = \dfrac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ x + c_2 = \dfrac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \end{cases} \rightarrow (x+c_2)^2 + (y+c_1)^2 = \lambda^2 \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2+\dot{y}^2} = \lambda^2$$

take $\lambda > 0$: $2\pi\lambda = L \rightarrow \lambda = L/2\pi$ $\quad A = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$

Isoperimetric Inequality: A: area enclosed by the curve

$$A \le L^2/4\pi$$

$$J(\vec{y}) = \int_a^b f(t, \vec{y}, \dot{\vec{y}})\,dt$$

E-L eqn. $(f_{\dot{y}_i})' = f_{y_i}$ $\quad 1 \le i \le n$

Proposition: If $f = f(\vec{y}, \dot{\vec{y}})$ then

$$H = \sum_{i=1}^n \dot{y}_i f_{\dot{y}_i} - f$$

is constant on trajectories.

Proof: $$\frac{d}{dt} H = \sum_{i=1}^n \left(\ddot{y}_i \cancel{f_{\dot{y}_i}} + \dot{y}_i (f_{\dot{y}_i})'\right) - \sum_{i=1}^n \left(f_{y_i} \dot{y}_i + \cancel{f_{\dot{y}_i} \ddot{y}_i}\right) = 0$$

$= $ by E-L eqn

Hamilton's Principle

Motion of a point mass of mass m

Conservative force $\vec{F}$: $\vec{F} = -\nabla U$, U is a potential for $\vec{F}$,

U is the potential energy

$L = T - U$, T is kinetic energy, L is the Lagrangian

Action: $$A(\vec{y}) = \int_a^b L(t, \vec{y}, \dot{\vec{y}})\,dt$$