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## Integrals with Higher Derivatives

$$F(y) = \int_a^b f(x,y,y',y'')\,dx \qquad f = f(x,y,z,\omega),\ \ z = y',\ \omega = y''$$

$f \in C^1([a,b]\times\mathbb{R}^3)$

$y \in \mathcal{D} = \{y \in C^2[a,b];\ y(a)=a_0,\ y'(a)=a_1,\ y(b)=b_0,\ y'(b)=b_1\}$

$v \in \mathcal{D}_0 = \{y \in C^2[a,b];\ y(a)=y'(a)=y(b)=y'(b)=0\}$

Suppose $y_0$ is a local max or min of $F(y)$ on $\mathcal{D}$. $\quad y_0+\varepsilon v \in \mathcal{D}$

Let $\quad g(\varepsilon) = F(y_0+\varepsilon v)$

$g$ has a max or min at $\varepsilon = 0 \rightarrow g'(0)=0$

$$0 = g'(0) = \delta F(y_0;v) \overset{A1}{=} \int_a^b \left(f_y v + f_z v' + f_\omega v''\right)dx \quad \forall v \in \mathcal{D}_0$$

($y_0$ is a stationary or extremal function)

$$0 = \int_a^b (f_y v + f_z v' + f_\omega v'')\,dx = \int_a^b \left(f_y v - (f_z)'v - (f_\omega)'v'\right)dx + \cancel{f_z v\Big|_a^b}^{\,0} + \cancel{f_\omega v'\Big|_a^b}^{\,0}$$

$$= \int_a^b \left(f_y - (f_z)' + (f_\omega)''\right)v\,dx - \cancel{(f_\omega)'v\Big|_a^b}^{\,0} \qquad \forall v \in \mathcal{D}_0$$

## Variation of Fundamental Lemma

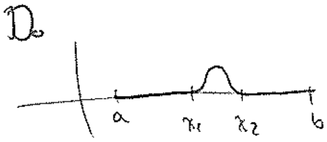
Let $h \in C^0[a,b]$, $\quad \int_a^b h(x)v(x)\,dx = 0 \quad \forall v \in \mathcal{D}_0$

Proof: Suppose $h(x_0) > 0$, $x_0 \in [a,b] \Rightarrow h(x) > 0$ on $[x_1,x_2] \subset [a,b]$, $x_0 \in [x_1,x_2]$. Let

$$v(x) = \begin{cases} (x-x_1)^3(x_2-x)^3 & x \in [x_1,x_2], \\ 0 & \text{otherwise.} \end{cases}$$

$v \in C^2[a,b]$, $\quad v(a) = v'(a) = v(b) = v'(b) = 0 \rightarrow v \in \mathcal{D}_0$

$$0 = \int_a^b h(x)v(x)\,dx = \int_{x_1}^{x_2} h(x)v(x)\,dx > 0$$




since $h(x)v(x) > 0$ on $(x_1,x_2)$

A contradiction. Similarly if $h(x_0) < 0$. $\therefore h = 0$ on $[a,b]$. ∎

Theorem: If $y_0$ is an extremal function for $F$ on $\mathcal{D}$ then

$$\boxed{f_y[y_0] - (f_z[y_0])' + (f_\omega[y_0])'' = 0}$$

Euler-Lagrange equation

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Example $F(y) = \int_0^{\pi/2} \left((y'')^2 - y^2\right) dx$

$D = \{y \in C^2[0,\pi/2];\ y(0)=1,\ y'(0)=0,\ y(\pi/2)=0,\ y'(\pi/2)=-1\}$

Find all extremal functions of $F$ on $D$

$f = -y^2 + w^2$

E-L eqn $\quad -2y + (2y'')'' = 0 \to y'''' - y = 0$

$y = e^{rx} \to r^4 - 1 = 0 \to (r^2+1)(r^2-1) = 0$

$\to r = \pm i, \pm 1$

$y = c_1 \cos x + c_2 \sin x + c_3 \cosh x + c_4 \sinh x$

$y' = -c_1 \sin x + c_2 \cos x + c_3 \sinh x + c_4 \cosh x$

$$\begin{cases} 1 = y(0) = c_1 + c_3 \\ 0 = y'(0) = c_2 + c_4 \\ 0 = y(\pi/2) = c_2 + c_3 \cosh(\pi/2) + c_4 \sinh(\pi/2) \\ -1 = y'(\pi/2) = -c_1 + c_3 \sinh(\pi/2) + c_4 \cosh(\pi/2) \end{cases}$$

$$\begin{bmatrix} \cosh \pi/2 & \sinh \pi/2 - 1 \\ \sinh \pi/2 + 1 & \cosh \pi/2 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det = \cdots = 2 \neq 0$

$\to c_3 = c_4 = 0 \to c_1 = 1,\ c_2 = 0$

$\therefore y = \cos x$ is the only extremal function

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I. Minimize $F$ on $D_1 = \{y \in C^2[a,b];\ y(a) = a_0,\ y(b) = b_0\}$

Suppose $y_0$ is a minimizer. Let $v \in C^2[a,b]$, $v(a) = v(b) = 0$.

$\to y_0 + \varepsilon v \in D_1\ \forall \varepsilon \quad$ Let $g(\varepsilon) = F(y_0 + \varepsilon v)$

$\to g'(0) = 0$

$\to 0 = \delta F(y_0; v) = \int_a^b \left(f_y[y_0]v + f_z[y_0]v' + f_w[y_0]v''\right) dx$

$= \int_a^b \left(f_y[y_0]v - (f_z[y_0])'v - (f_w[y_0])'v'\right) dx + f_z[y_0]v\Big|_a^b + f_w[y_0]v'\Big|_a^b$

$= \int_a^b \left(f_y[y_0] - (f_z[y_0])' + (f_w[y_0])''\right) v\, dx + \left(f_z[y_0] - (f_w[y_0])'\right)v\Big|_a^b + f_w[y_0]v'\Big|_a^b$

(i) Require $v(a) = v(b) = v'(a) = v'(b) = 0$.

$\int_a^b \left(f_y - (f_z)' + (f_w)''\right) v\, dx = 0$

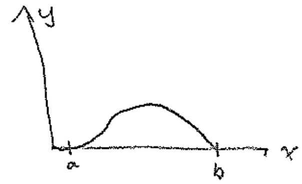
Variant of Fundamental lemma $\Rightarrow f_y[y_0] - (f_z[y_0])' + (f_w[y_0])'' = 0$

(i) $\left(f_z[y_0] - (f_w[y_0])'\right)v\Big|_a^b + f_w[y_0]v'\Big|_a^b = 0$

$v(a) = v(b) = 0 \longrightarrow f_w[y_0]v'\Big|_a^b = 0$ for all $v \in C^2[a,b]$ st $v(a)=v(b)=0$

$\exists v \in C^2[a,b]$ st $v(a) = v(b) = v'(a) = 0,\ v'(b) \neq 0 \Rightarrow f_w[y_0(b)] = 0$

Take $v(x) = (x-a)^2(x-b)$




$\exists v \in C^2[a,b]$ st $v(a) = v(b) = v'(b) = 0,\ v'(a) \neq 0$

$\Rightarrow f_w[y_0(a)] = 0$

Take $v(x) = (x-a)(x-b)^2$

Two natural boundary conditions:

$$f_w[y_0(a)] = 0, \quad f_w[y_0(b)] = 0$$

Ex Minimize $\quad F(y) = \int_0^1 \left((y')^2 + (y'')^2\right) dx$

on $\quad D_1 = \{y \in C^2[0,1] \ ; \ y(0) = 0,\ y(1) = 1\}$

E-L eqn: $f_y - (f_z)' + (f_w)'' = 0$

$-(2y')' + (2y'')'' = 0$

$\rightarrow y'''' - y'' = 0$

$y(0) = 0,\ y(1) = 1,$ Natural BCs: $f_w[y]\big|_{x=0} = 0 \quad f_w[y]\big|_{x=1} = 0$

$\rightarrow 2y''(0) = 0 \quad \rightarrow 2y''(1) = 0$

$\rightarrow y''(0) = y''(1) = 0$

$y = e^{rx} \rightarrow r^4 - r^2 = 0 \rightarrow r^2(r^2-1) = 0 \rightarrow r = 0, 0, 1, -1$

$y = C_1 + C_2 x + C_3 \cosh x + C_4 \sinh x \rightarrow y'' = C_3 \cosh x + C_4 \sinh x$

$y(0) = 0$: $C_1 + C_3 = 0$

$y(1) = 1$: $C_1 + C_2 + C_3 \cosh 1 + C_4 \sinh 1 = 1$

$y''(0) = 0$: $C_3 = 0$

$y''(1) = 0$: $C_3 \cosh 1 + C_4 \sinh 1 = 0$

$\rightarrow C_4 = 0, \rightarrow C_1 = -C_3 = 0, \ C_2 = 1$

$y = x$ is the only stationary function

Claim: $F$ is strictly convex on $D_1$

Take $y, y+v \in D_1 \rightarrow v(0) = v(1) = 0$

$F(y+v) = \int_0^1 \left((y'+v')^2 + (y''+v'')^2\right) dx$

That is the wild and wonderful condition

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$$= \underbrace{\int_0^1 \left((y')^2 + (y'')^2\right) dx}_{F(y)} + \underbrace{\int_0^1 \left(2y'v' + 2y''v''\right) dx}_{\delta F(y;v)} + \underbrace{\int_0^1 \left((v')^2 + (v'')^2\right) dx}_{\geq 0}$$

$F(y+v) \geq F(y) + \delta F(y;v)$, equality only if $v' = 0 \iff v = 0$ since $v(0) = v(1) = 0$

$F$ is strictly convex on $\mathcal{D}_1$

$y = x$ is a stationary function $\to$ $y = x$ is the unique minimizer of $F$ on $\mathcal{D}_1$.

II. Minimize $F$ on $\mathcal{D}_2 = \{y \in C^2[a,b];\ y(a) = a_0, y'(a) = a_1\}$

Suppose $y_0$ is a minimizer. Take $v \in C^2[a,b]$, $v(a) = v'(a) = 0$.

$$0 = \delta F(y_0; v) = \int_a^b (f_y v + f_z v' + f_w v'') dx$$

$$= \int_a^b \left(f_y - (f_z)' + (f_w)''\right) v \, dx + \left(f_z - (f_w)'\right) v \Big|_a^b + f_w v' \Big|_a^b$$

(i) Take $v \in C^2[a,b]$ st $v(a) = v'(a) = v(b) = v'(b) = 0$.

$$\longrightarrow f_y - (f_z)' + (f_w)'' = 0$$

(ii) $\left(f_z - (f_w)'\right) v \Big|_a^b + f_w v' \Big|_a^b = 0$ provided $v(a) = v'(a) = 0$

Take $v$ st $v(a) = v(b) = v'(a) = 0$, $v'(b) \neq 0$

$\Longrightarrow \underline{f_w[y_0(b)] = 0}$

Take $v$ st $v(a) = v'(a) = v'(b) = 0$, $v(b) \neq 0$

$\Longrightarrow \underline{f_z[y_0(b)] = (f_w[y_0(b)])'}$

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Ex $\quad F(y) = \int_0^1 \left((y')^2 + (y'')^2\right) dx$

$$\mathcal{D}_2 = \{y \in C^2[0,1];\ y(0) = 0,\ y'(0) = 1\}$$

$f = z^2 + w^2$, $\quad -(2y')' + (2y'')'' = 0 \longrightarrow y'''' - y'' = 0$

saw before $y = c_1 + c_2 x + c_3 \cosh x + c_4 \sinh x$

Natural BCs: $\quad 2y''(1) = 0 \qquad\qquad \longrightarrow y''(1) = 0$

$\qquad \left(2y' - (2y'')'\right)\Big|_{x=1} = 0 \longrightarrow y'(1) - y'''(1) = 0$

$y' = c_2 + c_3 \sinh x + c_4 \cosh x$, $\ y'' = c_3 \cosh x + c_4 \sinh x$, $\ y''' = c_3 \sinh x + c_4 \cosh x$

| | | |
|---|---|---|
| $y(0) = 0$ | $c_1 + c_3 = 0$ | $\longrightarrow$ $c_4 = 1$, $c_3 = -\sinh 1 / \cosh 1$ |
| $y'(1) = 1$ | $c_2 + c_4 = 1$ | $c_1 = \sinh 1 / \cosh 1$ |
| $y''(1) = 0$ | $c_3 \cosh 1 + c_4 \sinh 1 = 0$ | |
| $y'(1) - y'''(1) = 0$ | $c_2 = 0$ | |

$$\Rightarrow \quad y = \frac{\sinh 1}{\cosh 1}\left(1 - \cosh x\right) + \sinh x$$

Claim $F$ is strictly convex on $\mathcal{D}_2$.

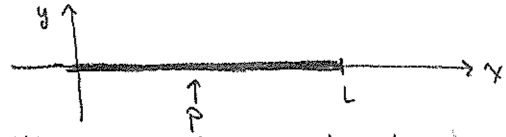
$$F(y+v) = F(y) + \delta F(y;v) + \int_0^1 \left((v')^2 + (v'')^2\right)dx$$

$$\Rightarrow \quad F(y+v) \geq F(y) + \delta F(y;v)$$

equality only if $v' = 0 \rightarrow v = c$, $v(0) = 0 \rightarrow c = 0$, $v = 0$

Claim is proved. $\therefore$ $y$ is the unique minimizer

## Elastic Beam




small deflection $y(x)$, $0 \leq x \leq L$,

load $p(x)$ (force per unit length, positive in positive $y$ direction)

$\mu > 0$ (flexural rigidity)

Clamped ends: $y(0) = y'(0) = y(L) = y'(L) = 0$

Potential energy: $$U(y) = \int_0^L \left(\frac{\mu}{2}(y'')^2 - py\right)dx$$

Minimize $U$ on $\mathcal{D} = \{y \in C^2[0,1],\ y(0) = y'(0) = y(L) = y'(L)\}$

Claim: $U$ is strictly convex on $\mathcal{D}$

Take $y, y+v \in \mathcal{D} \rightarrow v \in \mathcal{D}$

$$U(y+v) = \int_0^L \left(\frac{\mu}{2}(y''+v'')^2 - p(y+v)\right)dx$$

$$= \underbrace{\int_0^L \left(\frac{\mu}{2}(y'')^2 - py\right)dx}_{U(y)} + \underbrace{\int_0^L (\mu y'' v'' - pv)\,dx}_{\delta U(y;v)} + \int_0^L \frac{\mu}{2}(v'')^2 dx$$

$$U(y+v) \geq U(y) + \delta U(y;v)$$

equality only if $v'' = 0$ $\quad v = c_1 + c_2 x$. $\quad v(0) = v'(0) = 0 \rightarrow v = 0$

E-L eqn: $$f_y - (f_z)' + (f_w)'' = 0$$
$$-p + (\mu y'')'' = 0$$
$$\boxed{\mu y'''' = p}$$

uniform loading: $p$ constant $\rightarrow$ $y = \frac{p}{24\mu}x^4 + c_1 + c_2 x + c_3 x^2 + c_4 x^3$

(algebra) $\cdots$ $\quad y = \frac{p}{24\mu} x^2 (x-L)^2$ unique minimizer

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## Cantilever Beam

Min $U$ on $\mathcal{D}_2 = \{y \in C^2[0,1]\ ;\ y(0) = y'(0) = 0\}$

$p$ is constant (assume)

so again $y = \frac{p}{24\mu}x^4 + c_1 + c_2 x + c_3 x^2 + c_4 x^3$

Natural BCs: $f_w[y]\big|_{x=L} = 0 \longrightarrow \mu y''(L) = 0 \rightarrow y''(L) = 0$

$\left(f_y[y] - (f_w[y])'\right)\big|_{x=L} = 0 \rightarrow -(\mu y)''\big|_{L=0} = 0 \rightarrow y'''(L) = 0$

(algebra) . . .

$y = \frac{p}{24\mu}x^2(x^2 - 4Lx + 6L^2)$

$y(L) = \frac{p}{\mu 8}L^4$

midterm up to here