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Ex Brachistochrone

$$f = \sqrt{\frac{1+(y')^2}{y}} \rightarrow f(y,z) = \sqrt{\frac{1+z^2}{y}} \quad f_z = \frac{z}{\sqrt{y}\sqrt{1+z^2}}$$

$$\cancel{y'\frac{z}{\sqrt{y}\sqrt{1+z^2}}\Big|_{z=y'} - \frac{1}{2}y^{-3/2}\sqrt{1+z^2}\Big|_{z=y'} = c_1}$$

$$\cancel{\frac{(y')^2}{\sqrt{y}\sqrt{1+(y')^2}} - \frac{1}{2y^{3/2}}\sqrt{1+(y')^2} = c_1}$$

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redo correctly

$$y'f_z(y,y') - f(y,y') = -c$$

$$\rightarrow \frac{y'}{\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}} - \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = -c$$

$$\rightarrow \frac{(y')^2 - (1+(y')^2)}{\sqrt{y}\sqrt{1+(y')^2}} = -c$$

$$\rightarrow \frac{-1}{\sqrt{y}\sqrt{1+(y')^2}} = -c$$

$$\rightarrow \sqrt{y}\sqrt{1+(y')^2} = \frac{1}{c} = c_1$$

$$\rightarrow y' = \frac{\sqrt{c_1^2 - y}}{\sqrt{y}}$$

(equivalent to what we had before)

Free End-Point Problem

$$J(y) = \int_a^b f(x,y,y')\,dx$$

$$D_1 = \{y \in C^1[a,b];\ y(a) = y_a\}$$
$$D_2 = \{y \in C^1[a,b];\ y(b) = y_b\}$$
$$D_3 = C^1[a,b]$$

Minimize $J(y)$ on D_1, D_2, or D_3.

Minimize $J(y)$ on D_1:
Suppose $y_0 \in D_1$ gives the min
Let $v \in C^1[a,b]$ such that $v(a) = 0$.

Then $y_0 + \alpha v \in D_1 \quad \forall \alpha \in \mathbb{R}$

Let $g(\alpha) = J(y_0 + \alpha v) \quad g: \mathbb{R} \to \mathbb{R}$

$\alpha = 0$ gives the min value of $g \implies g'(0) = 0$.

$$0 = g'(0) = \delta J(y_0; v) = \int_a^b \left(f_y[y_0] v + f_z[y_0] v' \right) dx$$

$$= \int_a^b \left(f_y[y_0] - (f_z[y_0])' \right) v \, dx + f_z[y_0] v \Big|_a^b$$

$$= \int_a^b \left(f_y[y_0] - (f_z[y_0])' \right) v \, dx + f_z[y_0(b)] v(b)$$

$\forall v \in C'[a,b]$ st $v(a) = 0$.

(1) Consider $v \in C'[a,b]$ st $v(a) = v(b) = 0$

$$\int_a^b \left(f_y[y_0] - (f_z[y_0])' \right) v \, dx = 0$$

Fundamental lemma $\implies f_y[y_0] - (f_z[y_0])' = 0$, E-L eqn.

(2) $f_z[y_0(b)] v(b) = 0 \quad \forall v \in C'[a,b]$ st $v(a) = 0$

Let $v = x - a \implies f_z[y_0(b)](b-a) = 0 \implies \boxed{f_z[y_0(b)] = 0}$

Natural Boundary Condition

Theorem: (1) If y_0 minimizes J on D_1 then y_0 satisfies the E-L equation and $f_z[y_0(b)] = 0$.

(2) If y_0 minimizes J on D_2 then y_0 satisfies the E-L equation and $f_z[y_0(a)] = 0$.

(3) If y_0 minimizes J on D_3 then y_0 satisfies the E-L equation and $f_z[y(a)] = 0$, $f_z[y(b)] = 0$.

Theorem: Suppose f is [strongly] convex with respect to (y,z) & $(f_z[y_0])' = f_y[y_0]$

(1) If $y_0(a) = y_a$, $f_z[y(b)] = 0$ then y_0 minimizes J [uniquely] on D_1

(2) If $y_0(b) = y_b$, $f_z[y(a)] = 0$ then y_0 minimizes J [uniquely] on D_2

(3) If $f_z[y_0(a)] = 0$, $f_z[y_0(b)] = 0$ then y_0 minimizes J [uniquely up to an additive constant or uniquely] on D_3.

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Proof: $J(y_0+v) \ge J(y_0) + \delta J(y_0; v) = J(y_0)$ —— ①

$\forall v \in C^1[a,b]$ So in (1), $v(a)=0$, in (2) $v(b)=0$

$f(x, y+v, y'+v') \ge f(x,y,y') + f_y(x,y,y')v + f_z(x,y,y')$ —— ②

If f is strongly convex wrt (y,z) equality holds in ②

$\Rightarrow v=0$ or $v'=0 \Rightarrow vv'=0 \Rightarrow (v^2/2)'=0$

In case (1), $v(a)=0 \Rightarrow v=0$

In case (2), $v(b)=0 \Rightarrow v=0$

In case (3) $v=$ const. $=c$

$J(y_0+c) = J(y_0)$

c may be forced to be 0

Equality in ① is equivalent to equality in ② ■

Example $J(y) = \int_1^2 \frac{(y')^2}{x} dx$

Minimize J on $D_2 = \{y \in C^1[1,2]; y(2)=3\}$

$f(x,z) = \frac{z^2}{x}$, $f_{zz} = \frac{2}{x} > 0$ f is strictly convex wrt z

$\rightarrow$ f is strongly convex wrt (y,z)

E-L eqn: $(f_z)' = f_y$

$\left(\frac{2y'}{x}\right)' = 0 \rightarrow \frac{2y'}{x} = c_1 \Rightarrow y = \frac{c_1}{4}x^2 + c_2$

nat. bd. cond: $f_z[y(1)] = 0$

$\rightarrow \frac{2y'(1)}{1} = 0 \rightarrow y'(1) = 0 \rightarrow c_1 = 0 \rightarrow y = c_2$

$\boxed{y=3}$ is the unique minimizer on D_2

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Minimize the above on $C^1[1,2]$

Again, $y = \frac{c_1}{4}x^2 + c_2$

nat. bd. cond: $f_z[y(1)] = 0$, $f_y[y(1)] = 0$

$\rightarrow y'(1) = 0$ $\rightarrow y'(2) = 0$

$y' = \frac{c_1}{2}x \rightarrow y' = 0 \rightarrow y$ constant

$J(c) = 0$ $\forall c$

Minimizer is unique up to an additive constant

Example $J(y) = \int_1^2 \frac{y^2}{x} dx$

Minimize J on $C'[1,2]$

$f = f(x,y) = y^2/x$ f strongly convex w.r.t. (y,z)

E-L eq'n $(f_z[y])' = f_y \to 2y/x = 0 \to y = 0$

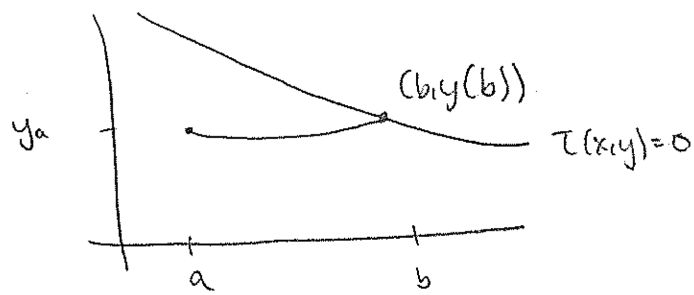
$y=0$ is the unique minimizer.

Variable End-Point Problems

Minimize $J(y,b) = \int_a^b f(x,y,y')dx$

subject to $y(a) = y_a$ & $\tau(b, y(b)) = 0$, $b > a$

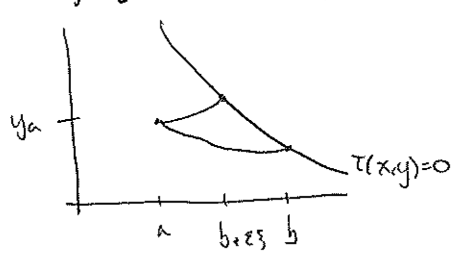
τ is a C' function such that $\nabla\tau = (\tau_x, \tau_y) \neq (0,0)$



Suppose (y_0, b_0) is a minimizer of

Consider $h(\varepsilon) = J(y_0 + \varepsilon v, b_0 + \varepsilon \xi)$

$v = v(x)$, ξ is a const. where $\tau(b_0 + \varepsilon\xi, y_0(b_0 + \varepsilon\xi) + \varepsilon v(b_0 + \varepsilon\xi)) = 0$ – (1)



h has a min at $\varepsilon = 0 \to h'(0) = 0$

$h(\varepsilon) = \int_a^{b+\varepsilon\xi} f(x, y_0 + \varepsilon v, y_0' + \varepsilon v')\,dx$

$0 = h'(0) \underset{\uparrow}{=} f[y(b_0)]\xi + \int_a^{b_0} \{f_y[y_0]v + f_z[y_0]v'\}dx$

Leibniz' rule

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For $\xi=0$: $\int_a^{b_0} \{ f_y[y_0]v' + f[y_0]v' \} dx = 0$

$v(b)=0, v(a)=0 \Rightarrow \int_a^{b_0} \{ f_y[y_0] - (f_z[y])' \} v\, dx = 0$

$\Rightarrow f_y[y_0] = (f_z[y_0])' = 0$ E-L eqn

For $\xi \neq 0$: $0 = \int_a^{b_0} \underbrace{\{ f_y[y_0] - (f_z[y_0])' \}}_{=0} v\,dx + f_z[y_0(b_0)]\, v(b_0) + f[y_0(b_0)]\,\xi$

$v(a)=0$

$$f[y_0(b_0)] + f_z[y_0(b_0)]\, v(b_0) = 0 \quad \text{—— (2)}$$

$v(b_0)$ is determined by differentiating (1) wrt t & setting $\xi=0$:

$\tau_x(b_0, y_0(b_0))\xi + \tau_y(b_0, y_0(b_0))(y_0'(b_0)\xi + v(b_0)) = 0$

$\Rightarrow v(b_0) = \xi\left(-y_0'(b_0) - \dfrac{\tau_x(b_0, y_0(b_0))}{\tau_y(b_0, y_0(b_0))} \right)$

Put this into (2):

$\tau_y(b_0, y_0(b_0)) f[y_0(b_0)] = f_z[y_0(b_0)]\left(\tau_y(b_0, y_0(b_0)) y'(b_0) + \tau_x(b_0, y_0(b_0)) \right)$

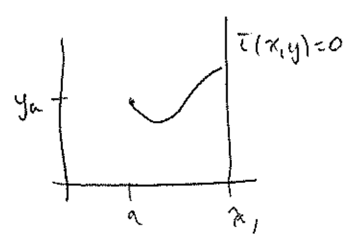
or $\boxed{\tau_y f = f_z(\tau_y y' + \tau_x)}$

[Transversality Condition]

Special cases:

(a) $\tau(x,y) = x - x_1$

$f_z[y_0(x_1)] = 0$

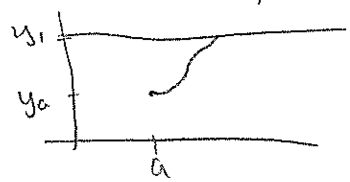


(b) $\tau(x,y) = y - y_1$

$\tau_y = 1 \quad \tau_x = 0$

$y_0'(b_0) f_z[y_0(b_0)] - f[y_0(b_0)] = 0$

$y' f_z - f = 0$ Hamiltonian



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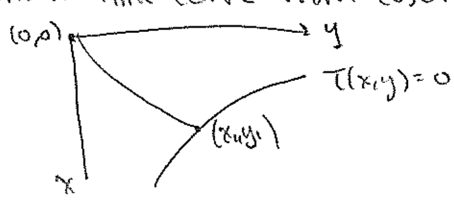
Ex Brachistochrone

Seek minimum time curve from $(0,0)$ to (x_1, y_1), $\tau(x_1, y_1) = 0$



$f = \frac{\sqrt{1+z^2}}{\sqrt{x}}$ $\qquad f_z = \frac{z}{\sqrt{x}\sqrt{1+z^2}}$, $z = y'$

$$\tau_y(x_1,y_1)\frac{\sqrt{1+y'(x_1)^2}}{\cancel{\sqrt{x_1}}} = \frac{y'(x_1)}{\cancel{\sqrt{x_1}}\sqrt{1+y'(x_1)^2}}$$

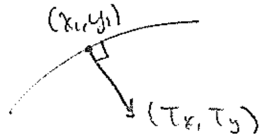
$$\tau_y(1+(y')^2) = \tau_y(y')^2 + y'\tau_x \longrightarrow \tau_y = y'\tau_x \text{ at } (x,y) = (x_1,y_1)$$

$(\tau_x, \tau_y)\cdot(y', -1) = 0$

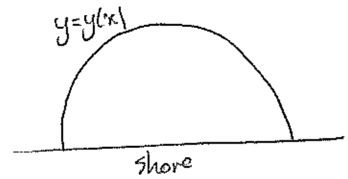
(τ_x, τ_y) is orthogonal to $\tau(x,y) = 0$

$(x, y(x))$ has tangent $(1, y')$

$(y', -1)$ is orthogonal to $(x, y(x)) \Rightarrow$ the curve $y = y(x)$ is orthogonal to $\tau(x,y) = 0$ at (x_1, y_1)



Dido's Problem



$y = y(x)$, $-a \le x \le a$, $a > 0$

$y(x) \ge 0$, $y \in C^1(-a,a) \cap C^0[-a,a]$

Maximize $\int_{-a}^{a} y(x)\,dx$ subject to $\int_{-a}^{a}\sqrt{1+y'(x)^2}\,dx = L$

$$\tau_y f = f_z(\tau_y y' + \tau_x) \quad \text{at } x = b \longleftarrow ?$$

Minimize $J(y,a) = -\int_{-a}^{a} y\,dx$

subject to $G(y,a) = \int_{-a}^{a}\sqrt{1+(y')^2}\,dx = L$

$f = -y$, $g = \sqrt{1+z^2}$

$\tilde{f} = f + \lambda g = -y + \lambda\sqrt{1+z^2}$

E-L eqn for $\tilde{f}$: $(\tilde{f}_z[y])' = \tilde{f}_y[y]$

$$\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right)' = -1 \longrightarrow \text{aside} \longrightarrow \begin{cases} \dfrac{y''}{(1+(y')^2)^{3/2}} = -\dfrac{1}{\lambda} \\ \text{curvature} = -1/\lambda \end{cases}$$

$$\frac{\lambda y'}{\sqrt{1+(y')^2}} = -x + C$$

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For $\lambda > 0$, $\tilde{f}$ is strictly convex wrt (y, z)
Expect unique minimizer. If $y(x)$ is a minimizer then $y(-x)$ is also a minimizer $\Rightarrow y(x) = y(-x) \Rightarrow y'(0) = 0 \Rightarrow c = 0$

$$\frac{\lambda y'}{\sqrt{1+(y')^2}} = -x$$

For $y(x) > 0$, $x > 0$, $y(a) = 0 \longrightarrow y'(x) < 0 \longrightarrow \lambda > 0$

$$\frac{(y')^2}{1+(y')^2} = \left(\frac{x}{\lambda}\right)^2 \longrightarrow y' = \pm \frac{x}{\sqrt{\lambda^2 - x^2}}$$

$y' < 0$ for $x > 0$ implies $y' = -x/\sqrt{\lambda^2 - x^2}$

$$\longrightarrow y = \sqrt{\lambda^2 - x^2} + d$$

Transversality condition
$T(x,y) = y$ endpoints are on $T(x,y) = 0$, $T_y = 1$, $T_x = 0$

$\tilde{f} = \tilde{f}_z y'$ or $y' \tilde{f}_z - \tilde{f} = 0$ at $x = \pm a$

$$\frac{y' \lambda y'}{\sqrt{1+(y')^2}} - \left(-y + \sqrt{1+(y')^2}\right) = 0$$

$$\frac{\lambda (y')^2}{\sqrt{1+(y')^2}} - \sqrt{1+(y')^2} + y = 0$$

$$\left(\frac{-\lambda}{\sqrt{1+(y')^2}} + y\right)\Bigg|_{x=\pm a} = 0 \longrightarrow \lim_{x \to a^-} |y'| = \infty$$

$$y' = \frac{-x}{\sqrt{\lambda^2 - x^2}} \qquad |y'| \to \infty \text{ as } x \to \pm a \Rightarrow \lambda = a$$

$$y = \sqrt{a^2 - x^2} + d$$

$y(\pm a) = 0 \Rightarrow d = 0$. Then $y = \sqrt{a^2 - x^2}$

Constraint: $\pi a = L \Rightarrow a = L/\pi$

$$\therefore \quad y = \sqrt{\left(\frac{L}{\pi}\right)^2 - x^2}$$