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Constraints

Theorem: Let $f, g \in C^0[[a,b] \times \mathbb{R}^2]$.

$$J(y) = \int_a^b f(x, y, y')\,dx$$

$$G(y) = \int_a^b g(x, y, y')\,dx$$

for $y \in D = \{v \in C^1[a,b];\ y(a) = y_0,\ y(b) = y_1\}$

Let $\tilde{f} = f + \lambda g$, λ a constant,

$$\tilde{J}(y) = \int_a^b \tilde{f}(x, y, y')\,dx.$$

Suppose y_0 minimizes $\tilde{J}$ on D [uniquely] Then y_0 minimizes J [uniquely] on $D_G = \{y \in D;\ G(y) = G(y_0)\}$.

Proof: $\tilde{J}(y) = J(y) + \lambda G(y) \geq \tilde{J}(y_0) = J(y_0) + \lambda G(y_0)$

For $y \in D_G$, $J(y) \geq J(y_0)$ So y_0 minimizes J on D.

Suppose y_0 minimizes $\tilde{J}$ uniquely on D.

If $J(y) = J(y_0)$, $y_0 \in D_G$

$\Rightarrow \tilde{J}(y) = \tilde{J}(y_0) \Rightarrow y = y_0$

$\therefore$ y min J uniquely on D_G. ☒

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Theorem: Let $f, g \in C^1([a,b] \times \mathbb{R}^2)$. f & λg convex wrt (y, z), λ a constant. [f or λg is strongly convex wrt (y,z)] $\tilde{f} = f + \lambda g$, $D = \{y \in C^1[a,b];\ y(a) = y_a,\ y(b) = y_b\}$, $y_0 \in D$ a solution to $(\tilde{f}_z[y_0])' = \tilde{f}_y[y_0]$. Then y_0 minimizes J [uniquely] on $D_G = \{y \in D;\ G(y) = G(y_0)\}$

Proof: $\tilde{f}$ is [strongly] convex wrt (y, z)

$\Rightarrow y_0$ minimizes $\tilde{J}$ [uniquely] on D

$\overset{\text{prev. thm}}{\Longrightarrow} y_0$ minimizes J [uniquely] on D_G

Example: $J(y) = \int_0^1 (y')^2\,dx$, $D = \{y \in C^1[0,1];\ y(0) = 0,\ y(1) = 1\}$

Minimize J on D subject to $\int_0^1 y\,dx = 1$.

$f = f(z) = z^2, \quad z = y'$

$g = g(y) = y$

$\tilde{f} = f + \lambda g = z^2 + \lambda y \qquad \lambda$ a constant

E-L eqn.: $\quad (\tilde{f}_z[y])' = \tilde{f}_y[y]$

$\tilde{f}_z = 2z \qquad (2y')' = \lambda \longrightarrow y'' = \frac{\lambda}{2}$

$\tilde{f}_y = \lambda$

$\longrightarrow y' = \frac{\lambda}{2}x + c_1 \longrightarrow y = \frac{\lambda}{4}x^2 + c_1 x + c_2$

let $\mu = \lambda/4 \qquad y = \mu x^2 + c_1 x + c_2$

$y(0) = 0 \longrightarrow c_2 = 0$

$y(1) = 1 \longrightarrow c_1 = 1 - \mu$

Thus $\quad y = \mu x^2 + (1-\mu)x$

$\int_0^1 y\,dx = 1 : \left(\mu \frac{x^3}{3} + (1-\mu)\frac{x^2}{2}\right)\Big|_0^1 = \frac{\mu}{3} + \frac{(1-\mu)}{2} = 1$

$\longrightarrow -\frac{\mu}{6} = \frac{1}{2} \longrightarrow \mu = -3$

$\therefore \; y_0 = -3x^2 + 4x$

$f = z^2, \; \lambda g = \lambda y$ are convex

f is strongly convex wrt (y, z)

$\therefore y_0$ is the unique minimizer of J st $\int_0^1 y\,dx = 1$

Special cases of the E-L eqns

$f(x, y, z), \quad z = y'; \qquad J(y) = \int_a^b f(x, y, y')\,dx$

No z $\quad f_y[y] = 0 \longrightarrow f_y(x, y) = 0$

Not an ODE. Solve for y

No y $\quad (f_y[y])' = 0 \longrightarrow f_z[y] = C_1$

$\longrightarrow f_z(x, y') = C_1 \quad 1^{st}$ order ODE

No x $\quad (f_z(y, y'))' = f_y(y, y')$

$f_{zy}y' + f_{zz}y'' = f_y(y, y')$, a 2^{nd} order ODE

Consider $\; y' f_z(y, y') - f(y, y') \quad$ (Hamiltonian)

$(y' f_z(y, y') - f(y, y'))' = y'' f_z + y'\underbrace{(f_{zy}y' + f_{zz}y'')}_{f_y \text{ by EL}} - (f_y y' - f_z y'') = 0$

Thus $\; y' f_z(y, y') - f(y, y') = c_1$

a 1^{st} order ODE