you're in charge of
the constants, don't
let them push you around

2015 10 07

Example: Minimize $J(y) = \int_0^1 ((y')^2 - xy)\,dx$

on $\mathcal{D} = \{ y \in C^1[0,1];\ y(0)=0,\ y(1)=1 \}$

$f(x,y,z) = z^2 - xy \longrightarrow f_z = 2z,\ f_y = -x$

E-L eqn $\quad (f_z[y(x)])' = f_y[y(x)]$

$\longrightarrow \quad (2y')' = -x$

$\longrightarrow \quad 2y' = -x^2/2 + 2c_1$

$\longrightarrow \quad y' = -x^2/4 + c_1$

$\quad y = -x^3/12 + c_1 x + c_2$

$y(0)=0 \longrightarrow c_2 = 0, \quad y(1) = 1 \rightarrow c_1 - 1/12 = 1 \rightarrow c_2 = 13/12$

$\therefore\ y = -x^3/12 + 13/12\, x$

$f = f_1 + f_2,\ f_1 = z^2,\ f_2 = -xy \qquad f_{zz} = 2 > 0$ ~~f_1 is strictly conv~~

$\longrightarrow f_1 = f_1(z)$ is strictly convex wrt z $\rightarrow f_1$ is stongly convex wrt (y,z)

$\nabla^2_{(y,z)} f_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is pos. semi-def. $\rightarrow f_2$ is convex wrt (y,z)

$\longrightarrow f = f_1 + f_2$ is strongly convex wrt (y,z)

$\rightarrow J(y)$ is strictly convex on $\mathcal{D}$.

$\therefore\ y = -x^3/12 + \frac{13}{12}x$ is the unique minimizer of J on $\mathcal{D}$

Brachistochrone Problem

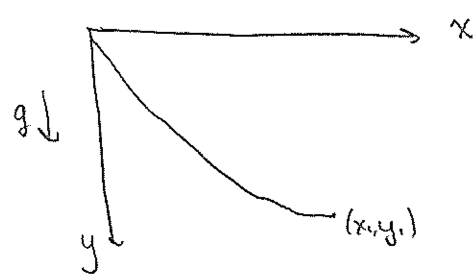
$x_1, y_1 > 0$

Minimize $J(y) = \frac{1}{\sqrt{2g}} \int_0^{x_1} \sqrt{\frac{1+(y')^2}{y}}\,dx$

on $\mathcal{D} = \{ y \in C^1(0,x_1] \cap C^0[0,x_1];$

$y(x) > 0$ on $(0, x_1]$,

$y(0)=0,\ y(x_1)=y_1\ \ \int_0^{x_1} \frac{dx}{\sqrt{y(x)}} < \infty \}$



Same as minimizing $J(y) = \int_0^{x_1} \sqrt{\frac{1+(y')^2}{y}}\,dx$ on $\mathcal{D}$

$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^{x_1} \sqrt{\frac{1+(y')^2}{y}}\,dx$

$f(y,z) = \frac{\sqrt{1+z^2}}{\sqrt{y}}$

for $y > 0,\ z \in \mathbb{R}$

$$\nabla^2_{(y,z)} f = \begin{bmatrix} \frac{3}{4}\frac{(1+z^2)^{1/2}}{y^{5/2}} & " \\ -\frac{1}{2} y^{-3/2} \frac{z}{\sqrt{1+z^2}} & \frac{1}{\sqrt{y}\,(1+z^2)^{3/2}} \end{bmatrix}$$

I'm confused by the notation I'm about to introduce

I've been calling all the functionals J, I don't think it's confusion

$a_{11}>0, \quad a_{22}>0$

$$a_{11}a_{22}-a_{12}^2 = \frac{1}{y^3}\frac{1}{(1+z^2)}\left(\frac{3}{4}-z^2\right)$$

$\rightarrow$ f is not convex wrt (y, z)

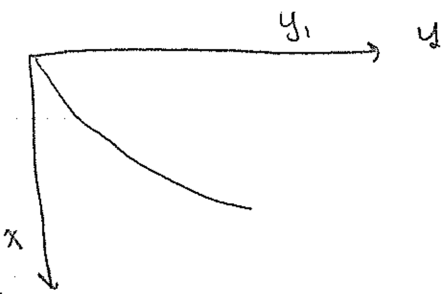
Suppose $x = x(y)$

$$J(y) = \int_0^{x_1}\sqrt{\frac{1+(dy/dx)^2}{y}}\,dx = \int_0^{y_1}\frac{\sqrt{1+(dy/dx)^2}}{\sqrt{y}}\frac{dx}{dy}\,dy$$

$$= \int_0^{y_1}\frac{\sqrt{(dx/dy)^2+1}}{\sqrt{y}}\,dy = \bar{J}(x)$$

Interchange x & y, x_1 & y_1

$$J(y) = \int_0^{x_1}\frac{\sqrt{1+(dy/dx)^2}}{\sqrt{x}}\,dx$$



on $\mathcal{D} = \{y \in C^1[0,x_1];\ y(0)=0,\ y(x_1)=y_1\}$

$$J(y) = \lim_{\varepsilon\to 0^+}\int_\varepsilon^{x_1}\frac{\sqrt{1+(y')^2}}{\sqrt{x}}\,dx$$

exists since $\displaystyle\int_\varepsilon^{x_1}\frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_\varepsilon^{x_1} \longrightarrow 2\sqrt{x_1}$ as $\varepsilon\to 0^+$

$$f = f(x,z) = \frac{\sqrt{1+z^2}}{\sqrt{x}} \qquad f_{zz} = \frac{1}{\sqrt{x}}\frac{1}{(1+z^2)^{3/2}} > 0 \quad \forall z, \forall x>0$$

E-L eqn: $\quad \left(f_z[y(x)]\right)' = f_y[y(x)]$

$$\left(\frac{1}{\sqrt{x}}\frac{z}{\sqrt{1+z^2}}\Big|_{z=y'}\right)' = 0 \rightarrow \left(\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+(y')^2}}\right)' = 0$$

$$\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{x}\sqrt{1+(y')^2}} = \frac{1}{c}$$

$$c^2(y')^2 = x(1+(y')^2)$$

$$\rightarrow (y')^2(c^2-x) = x$$

$$\rightarrow y' = \frac{\sqrt{x}}{\sqrt{c^2-x}}$$

$$x(\theta) = c^2\sin\theta/2 = \tfrac{c^2}{2}(1-\cos\theta), \quad c^2-x = \tfrac{c^2}{2}(1+\cos\theta)$$

motivation

$$\frac{dx}{d\theta} = \frac{c^2}{2}\sin\theta \qquad \frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = \frac{\sqrt{x}}{\sqrt{c^2-x}}\frac{c^2}{2}\sin\theta$$

$$= \frac{\frac{c}{\sqrt{2}}\sqrt{1-\cos\theta}}{\frac{c}{\sqrt{2}}\sqrt{1+\cos\theta}}\frac{c^2}{2}\sin\theta$$

2015 10 07(2)

$$= \frac{c^2}{2}(1-\cos\theta)$$

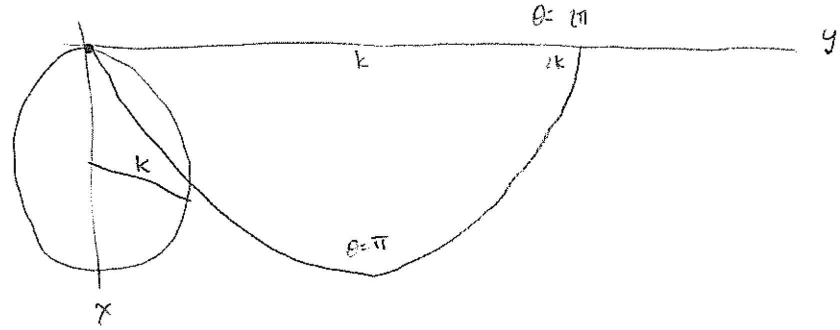
$$y(\theta) = \frac{c^2}{2}(\theta - \sin\theta) + c_1$$

$$x(0)=0 \rightarrow y(0)=0 \rightarrow c_1=0$$

Let $k = c^2/2$ $(k>0)$. Thus

$$x(\theta) = k(1-\cos\theta) \quad y(\theta) = k(\theta - \sin\theta)$$

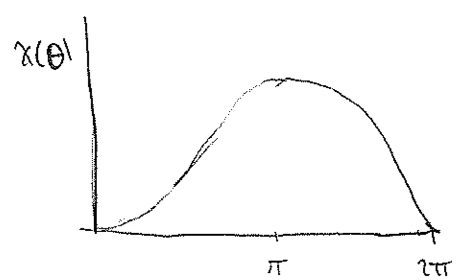
This is a cycloid

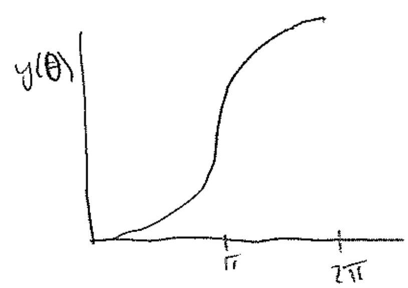


2015 10 09

$(x(0), y(0)) = (0,0)$,

obtained ~~by~~ from a circle of radius k rolling on the y-axis, starting with the circle tangent at (0,0)





$x'(\theta) > 0$ on $(0,\pi)$,

$\rightarrow x(\theta)$ is strictly increasing on $[0,\pi]$

$y'(\theta) > 0$ on $(0, 2\pi)$

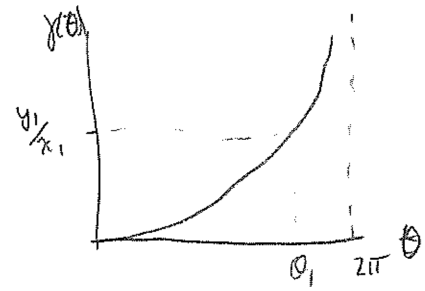
$\rightarrow y(\theta)$ is strictly increasing on $[0, 2\pi]$

$$\gamma(\theta) = \frac{y(\theta)}{x(\theta)} = \frac{\theta - \sin\theta}{1-\cos\theta} \qquad \lim_{\theta \to 0^+} \gamma(\theta) = 0 \quad \lim_{\theta \to 2\pi^-} \gamma(\theta) = \infty$$

$\gamma'(\theta) > 0$ on $(0, 2\pi)$

$\rightarrow \gamma(\theta)$ is strictly increasing on $[0, 2\pi]$

$(x(\theta), y(\theta)) = (x_1, y_1) \Rightarrow \gamma(\theta_1) = y_1/x_1$

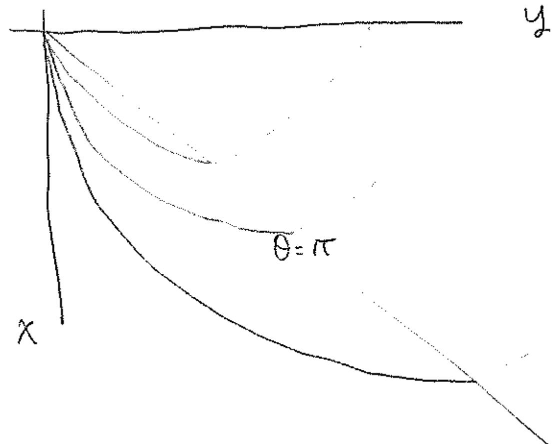


k is determined by $x_1 = x(\theta_1)$

$$= k(1-\cos\theta_1)$$

$$\rightarrow k = \frac{x_1}{1-\cos\theta_1}$$

$$\gamma(\pi) = \frac{\pi}{2} = \frac{y(\pi)}{x(\pi)}$$



If $y_1/x_1 < \pi/2$

$x(t)$ is strictly increasing on $[0, \theta_1]$

$\theta = h(x)$

$y = k(\theta - \sin\theta) = k(h(x) - \sin h(x))$

so $y = y(x)$

y satisfies E.L eqn

$$f(x, z) = \frac{\sqrt{1+z^2}}{\sqrt{x}}$$

Take $v \in C^1[0, x_1]$, $v(0) = v(x_1) = 0$

$$f(x, y'+v') \geq f(x, y') + f_z(x, y')v'$$

$$\int_{\varepsilon}^{x_1} f(x, y'+v')\,dx \geq \int_{\varepsilon}^{x_1} \left(f(x, y') + f_z(x, y')v'\right)dx$$

equality only for $v'=0$

$$= \int_{\varepsilon}^{x_1} \Big[f(x, y') - \underbrace{(f_z(x, y'))'}_{=0} v\Big]\,dx + f_z(x, y')v\Big|_{\varepsilon}^{x_1}$$

Let $\varepsilon \to 0$: $J(y+v) \geq J(y)$

equality only if $v' = 0$ on $[0, x_1]$

which occurs only if $v = 0$ on $[0, x_1]$

$\therefore$ y minimizes J on D uniquely