Variation or Gateaux Derivative

<u>Def</u> $\mathcal{D} \subset \mathcal{Y}$, $\mathcal{Y}$ a linear space of functions

$J : \mathcal{D} \to \mathbb{R}$

suppose $y, v \in \mathcal{Y}$ $y, y+v \in \mathcal{D}$
for $|\alpha|$ sufficiently small

$$\delta J(y;v) = \lim_{\alpha \to 0} \frac{J(y+\alpha v) - J(y)}{\alpha}$$

$$= \frac{d}{d\alpha} J(y+\alpha v)\Big|_{\alpha=0}$$

provided it exists.

Theorem: Let

$$F(\alpha) = \int_a^b f(x,\alpha)\,dx,$$

for $f \in C^1([a,b] \times I)$, I some interval. Then

$$F'(\alpha) = \int_a^b f_\alpha(x,\alpha)\,dx,$$

for $f_\alpha = \partial f / \partial \alpha$.

Proof: $$F'(\alpha) = \lim_{\varepsilon \to 0} \frac{F(\alpha+\varepsilon) - F(\alpha)}{\varepsilon}$$

$$F(\alpha+\varepsilon) - F(\alpha) = \int_a^b \left[f(x,\alpha+\varepsilon) - f(x,\alpha) \right] dx$$

$$= \int_a^b \left[\int_\alpha^{\alpha+\varepsilon} f_\alpha(x,s)\,ds \right] dx$$

$$\to \frac{F(\alpha+\varepsilon) - F(\alpha)}{\varepsilon} = \int_a^b \left[\frac{1}{\varepsilon} \int_\alpha^{\alpha+\varepsilon} f_\alpha(x,s)\,ds \right] dx$$

$$\longrightarrow \int_a^b f_\alpha(x,\alpha)\,dx \quad \text{as} \quad \varepsilon \to 0. \qquad \blacksquare$$

Theorem: Let

$$J(y) = \int_a^b f(x, y(x), y'(x))\,dx,$$

$f(x,y,z) \in C^1([a,b]\times\mathbb{R}\times\mathbb{R})$. Then for $y, v \in C^1[a,b]$,

$$\delta J(y;v) = \int_a^b \left(f_y[y(x)]v(x) + f_z[y(x)]v'(x) \right] dx$$

with $[y(x)] = (x, y(x), y'(x))$

Proof:

$$J(y+\alpha v) = \int_a^b f(x, y(x)+\alpha v(x), y'(x)+\alpha v'(x))\,dx$$

$$\frac{d}{d\alpha} J(y+\alpha v) = \int_a^b \Big\{ f_y(x, y(x)+\alpha v(x), y'(x)+\alpha v'(x))v(x)$$
$$+ f_z(x, y(x)+\alpha v(x), y'(x)+\alpha v'(x))v'(x) \Big\} dx$$

$$\Rightarrow \delta J(y;v) = \frac{d}{d\alpha} J(y+\alpha v)\Big|_{\alpha=0}$$
$$= \int_a^b$$

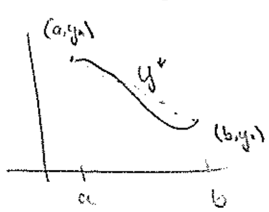
Ex. $J(y) = \int_a^b \sqrt{1+y'(x)^2}\,dx$ for $y \in C^1[a,b]$

[Note: $\delta J(y,v)$ is linear in v: $\delta J(y; v_1+v_2) = \delta J(y;v_1) + \delta J(y;v_2)$
$\delta J(y; cv) = c\,\delta J(y;v)$.

$$f(x,y,z) = g(z) = \sqrt{1+z^2}$$
$$f_z = g'(z) = \frac{z}{\sqrt{1+z^2}}$$

For $y, v \in C^1[a,b]$, $\delta J(y;v) = \int_a^b \frac{y'(x)}{\sqrt{1+y'(x)^2}} v'(x)\,dx$

Shortest Curve between Two Points



$J(y) = \int_a^b \sqrt{1+y'(x)^2}\,dx$ for $y \in \mathcal{D} = \{y \in C^1[a,b];\ y(a)=y_0,\ y(b)$

$y^* = m(x-a)+y_0,\quad m = \frac{y_1-y_0}{b-a}$

Claim: $J(y) \geq J(y^*)$ $\forall y \in \mathcal{D}$ & equality holds if and only if $y = y^*$

$$g(z) = \sqrt{1+z^2}$$
$$g'(z) = \frac{z}{\sqrt{1+z^2}} = z(1+z^2)^{-1/2}$$
$$g''(z) = (1+z^2)^{-1/2} + z\left(-\tfrac{1}{2}\right)(1+z^2)^{-3/2}\, 2z \overset{=\cdots=}{=} \frac{1}{(1+z^2)^{3/2}} > 0$$

$\Rightarrow g$ is strictly convex

$g(z) \geq g(m) + g'(m)(z-m)$ $\quad \forall z$

equality only when $z = m$

$\Rightarrow g(y'(x)) \geq g(m) + g'(m)\cdot(y'(x)-m)$, equality only if $y'(x) = m$ 2015 09 28

$$J(y) = \int_a^b g(y'(x))\,dx \geq \int_a^b [g(m) + g'(m)(y'(x)-m)]\,dx$$
$$= \int_a^b g(m)\,dx + g'(m)\left(y(b) - y(a) - m(b-a)\right)^{\to 0}$$
$$= J(y^*) \qquad (\text{since } y^{*\prime}(x) = m)$$

equality only if $y'(x) = m$ $\forall x \in [a,b]$

$\Rightarrow y(x) = m(x-a) + C$

$y(a) = y_0 \Rightarrow C = y_0$

$\rightarrow y(x) = m(x-a) + y_0 = y^*(x)$

$y^*(x)$ is the unique minimizer

Def] Let $\mathcal{D} \subseteq \mathcal{Y}$, $\mathcal{Y}$ a func space, $J: \mathcal{D} \to \mathbb{R}$

(1) $y^* \in \mathcal{D}$ minimizes J on $\mathcal{D}$ means
$$J(y) \geq J(y^*), \quad \forall y \in \mathcal{D}$$

(2) $y^* \in \mathcal{D}$ minimizes J uniquely on $\mathcal{D}$ means
$$J(y) \geq J(y^*) \ \forall y \in \mathcal{D}, \text{ equality holds only if } y = y^*$$

(3) $y^* \in \mathcal{D}$ is stationary for J means
$$\delta J(y^*; v) = 0 \quad \forall v \in \mathcal{Y} \text{ st } y+v \in \mathcal{D}$$

(4) J is convex on $\mathcal{D}$ means
$$J(y+v) \geq J(y) + \delta J(y; v) \quad \forall y, y+v \in \mathcal{D}$$

(5) J is strictly convex on $\mathcal{D}$ means

$J(y+v) \geq J(y) + \delta J(y; v)$ $\forall y, y+v \in \mathcal{D}$, equality holds only for $v = 0$.

Ex. (cont.) $J(y) = \int_a^b \sqrt{1+y'(x)^2}\, dx$

$\mathcal{Y} = C^1[a,b]$, $\mathcal{D} = \{y \in \mathcal{Y};\ y(a) = y_0,\ y(b) = y_1\}$

$$\delta J(y;v) = \int_a^b f_z[y(x)]\, v'(x)\, dx \qquad [y(x)] = (x, y(x), y'(x))$$

$$= \int_a^b \frac{y'(x)}{\sqrt{1+y'(x)^2}}\, v'(x)\, dx$$

$$\delta J(y^*;v) = \int_a^b \frac{m}{\sqrt{1+m^2}}\, v'(x)\, dx = \frac{m}{\sqrt{1+m^2}}[v(b) - v(a)]$$

$y, y+v \in \mathcal{D} \Rightarrow v(a) = v(b) = 0$

$\mathcal{D}_0 = \{v \in \mathcal{Y};\ v(a) = v(b) = 0\}$

y^* is stationary

Check convexity of J

$J(y+v) = \int_a^b g(y'+v')\, dx$

$g(y'+v') \geq g(y') + g'(y')v'$, equality holds only for $v' = 0$.

by the strict convexity of g

$\Rightarrow J(y+v) = \int_a^b g(y'+v')dx \geq \int_a^b g(y')dx + \int_a^b g'(y')v'\, dx$

$= J(y) + \delta J(y, v)$

J is convex on $\mathcal{Y}$. Show J is strictly convex on $\mathcal{D}$

$J(x+y) = J(y) + \delta J(y;v) \Rightarrow v' = 0$ on $[a,b]$

$v \in \mathcal{D}_0$: $v(a) = v(b) = 0 \Rightarrow v(x) = c$, $v(a) = 0 \Rightarrow c = 0$

$\therefore v(x) = 0$ on $[a,b]$. Thus J is str. convex on $\mathcal{D}$

Basic Problem: Minimize $J(y) = \int_a^b f(x, y(x), y'(x))\, dx$, $f \in C^1$

subject to $y(a) = y_0$, $y(b) = y_1$.

$\mathcal{Y} = C^1[a,b]$, $\mathcal{D} = \{y \in \mathcal{Y};\ y(a) = y_0,\ y(b) = y_1$

Minimize J on $\mathcal{D}$.

Theorem: If y^* minimizes J on $\mathcal{D}$ then y^* is stationary for J.

Proof: Let $g(\alpha) = J(y^* + \alpha v)$, $v \in \mathcal{D}_0 = \{y \in \mathcal{Y};\ y(a) = y(b) = 0\}$

Note $y^* + \alpha v \in \mathcal{D}\ \forall \alpha$, $g: \mathbb{R} \to \mathbb{R}$, $J(y^* + \alpha v) \geq J(y^*)$

$\Rightarrow g(\alpha) \geq g(0)\ \forall \alpha \Rightarrow g'(0) = 0 \Rightarrow \delta J(y^*; v) = 0$

y^* is stationary for J.

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$$\delta J(y;v) = \int_a^b \left\{ f_y[y(x)]\,v(x) + f_z[y(x)]\,v'(x) \right\} dx$$

Theorem:

(1) If y^* is stationary for J on D & J is convex on D then y^* minimizes J on D.

(2) If y^* is stationary for J on D & J is strictly convex on D then y^* minimizes J uniquely on D.

Proof: (1) $J(y^*+v) \geq J(y^*) + \delta J(y^*;v), \quad \forall v \in D_0$

$= J(y^*)$

$\Rightarrow$ for any $y \in D$, $J(y) \geq J(y^*)$

(2) As in (1), $J(y^*+v) \geq J(y^*)$, equality only if $v = 0$

$\Rightarrow$ For any $y \in D$, $J(y) \geq J(y^*)$, equality only if $y = y^*$ ■

Local max or min can be defined using a norm

On $C^0[a,b]$, $\|y\|_0 = \max_{a \leq x \leq b} |y(x)|$

On $C^1[a,b]$, $\|y\|_1 = \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} |y'(x)|$

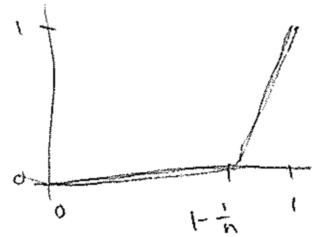
There may be no solution

Example. $J(y) = \int_0^1 y(x)^2 dx$ on $D = \{ y \in C^0[0,1] ; y(0) = 0, y(1) = 1 \}$.

$J(y) \geq 0$ and $J(y) = 0 \Rightarrow y = 0$ on $[0,1]$ but $y \notin D$

Let $y_n(t) = \begin{cases} 0 & ; 0 \leq x \leq 1 - \frac{1}{n} \\ n(x - (1 - 1/n)) & ; 1 - \frac{1}{n} \leq x \leq 1 \end{cases}$

for $n \geq 2$. Note $y_n \in D \ \forall n$



$$J(y_n) = \int_{1-\frac{1}{n}}^{1} n^2 \left(x - \left(1 - \tfrac{1}{n}\right)\right)^2 dx = \frac{n^2}{3} \left(x - \left(1 - \tfrac{1}{n}\right)\right)^3 \Bigg|_{1-\frac{1}{n}}^{1}$$

$$= \frac{n^2}{3} \frac{1}{n^3} = \frac{1}{3n}$$

$J(y_n) \longrightarrow 0$ as $n \to \infty$

$\therefore$ There is no function which minimizes J on D

$\left(\inf_{y \in D} J(y) = 0 \right)$

Convex & Strictly Convex Functionals

$$J(y) = \int_a^b f(x, y(x), y'(x))\,dx$$

$$\delta J(y;v) = \int_a^b \{ f_y[y(x)]v(x) + f_z[y(x)]v'(x) \}\,dx$$

$f(x,y,z)$ is convex with respect to (y,z):

$$f(x, y+v, z+w) \geq f(x,y,z) + f_y(x,y,z)v + f_z(x,y,z)w \quad — (1)$$

$$\forall v, w, \quad \forall x \in [a,b]$$

$f(x,y,z)$ is strictly convex with respect to (y,z): (1) holds and equality in (1) holds $\longleftrightarrow$ $v=0$ & $w=0$

$f(x,y,z)$ is strongly convex with respect to (y,z): (1) holds & equality in (1) holds ~~if~~ only if $v=0$ or $w=0$

Theorem: If $f(x,y,z)$ is strongly convex with respect to (y,z) $\forall x \in [a,b]$ then $J(y)$ is strictly convex on $\mathcal{D}$

Proof: $J(y+v) = \int_a^b f(x, y+v, y'+v')\,dx$ $\quad y \in \mathcal{D}, v \in \mathcal{D}_0$

$$\geq \int_a^b f(x,y,y') + f_y(x,y,y')v + f_z(x,y,y')v'\,dx$$

$$= J(y) + \delta J(y;v)$$

Equality holds when $v=0$ or $v'=0 \Rightarrow vv'=0$ $\forall x \rightarrow \left(\frac{v^2}{2}\right)' = 0$ $\forall x$ $\rightarrow v^2/2 = c$ $\quad v \in \mathcal{D}_0 \Rightarrow c=0 \quad \therefore v=0$

$\therefore J$ is strictly convex on $\mathcal{D}$.

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Theorem: $J(y) = \int_a^b f(x,y,y')\,dx$, $y \in \mathcal{D} = \{ y \in C^1[a,b] ; y(a)=y_0, y(b)=y_1 \}$

(1) f is convex wrt $(y,z) \Rightarrow J$ is convex on $\mathcal{D}$

(2) f is strongly convex wrt $(y,z) \Rightarrow J$ is strictly convex on $\mathcal{D}$

Proof: (2) last time

(1) $f(x, y+v, y'+v') \geq f(x,y,y') + f_y(x,y,y')v + f_z(x,y,y')v'$ $\quad y \in \mathcal{D}, v \in \mathcal{D}_0$

$\Rightarrow J(y+v) \geq J(y) + \delta J(y;v)$ $\quad$ (integrating $a \leq x \leq b$)

$\therefore J$ is convex on $\mathcal{D}$. $\quad \square$

Cor 1: If $f = f(x,z)$ is strictly convex wrt z then f is strongly convex wrt (y,z) & hence $J(y) = \int_a^b f(x,y')dx$ is strictly convex on D.

Proof: $f(x, z+w) \geq f(x,z) + f_z(x,z)w$ (1)
& equality holds in (1) $\Rightarrow w = 0$
Thus f is strongly convex wrt (y,z). Thm $\Rightarrow J$ is strictly convex. □

Cor 2: If $f = f(x,y)$ is strictly convex wrt y then f is strongly convex wrt (y,z), hence $J(y) = \int_a^b f(x,y)dx$ is ~~strongly~~ strictly? convex.

Proof: A2

Prop: If $f_1(x,y,z)$ is strongly convex wrt (y,z), $f_2(x,y,z)$ is convex wrt (y,z), $p(x) > 0$ on $[a,b]$, then $f_1(x,y,z) + f_2(x,y,z)$ is strongly convex wrt (y,z) & $p(x)f_1(x,y,z)$ is strongly convex wrt. (y,z).

Proof: A2.

Example $J(y) = \int_a^b ((y')^4 - xy)dx$.

$$D = \{y \in C^2[a,b];\ y(a) = y_0,\ y(b) = y_1\}$$

$$f(x,y,z) = z^4 - xy$$

$$\nabla^2_{(y,z)} f = \begin{pmatrix} f_{yy} & f_{yz} \\ f_{yz} & f_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 12z^2 \end{pmatrix}$$

f is convex wrt $(y,z) \Rightarrow J(y)$ is convex on D

$\nabla^2_{(y,z)} f$ is not pos. def.

Claim: f is strongly convex wrt (y,z)

$f = f_1 + f_2$, $f_1 = z^4$, $f_2 = -xy$

f_1 is strictly convex wrt z since $f_{zz} = 12z^2 > 0$ for $z \neq 0$

Cor 1 $\rightarrow$ f_1 is strongly convex wrt (y,z)

f_2 is convex wrt (y,z) since $\nabla^2_{(y,z)} f_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Prop $\rightarrow$ $f = f_1 + f_2$ is strongly convex wrt (y,z).

Thm $\Rightarrow$ $J(y)$ is strictly convex on D.

Example $J(y) = \int_a^b yy'\,dx \qquad D = \{y \in C'[a,b];\ y(a) = y_0,\ y(b) = y_1\}$

$f = f(y,z) = yz$

$\nabla^2_{(y,z)} f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, det $= -1 \to f$ is not convex wrt (y,z)

$J(y) = \int_a^b yy'\,dx = \int_a^b \left(y^2/2\right)'\,dx = y^2/2 \Big|_a^b = \dfrac{y^2(b) - y^2(a)}{2} = \dfrac{y_1^2 - y_0^2}{2}$

$J(y)$ is constant on D!

$\delta J(y,v) = \lim_{\alpha \to 0} \dfrac{J(y+\alpha v) - J(y)}{\alpha} = 0$

$J(y+v) = (y_1^2 - y_0^2)/2$

$J(y) + \delta J(y;v) = J(y)$

So $J(y+v) \geq J(y) + \delta J(y;v)$

$\forall y \in D,\ \forall y \in D_0$

$\therefore J$ is convex on D.

(yy' is a null Lagrangian)

2015 10 05

Euler-Lagrange Equation

Suppose $y \in D$ is stationary for $J(y) = \int_a^b f(x,y,y')\,dx$ on $D = \{y \in C'[a,b];\ y(a) = y_0,\ y(b) = y_1\}$.

$$\delta J(y;v) = \int_a^b \left\{ f_y[y(x)]v(x) + f_z[y(x)]v'(x) \right\} dx = 0$$

$$\forall v \in D_0 = \{v \in C'[a,b];\ v(a) = v(b) = 0\}$$

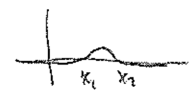
$$0 = \int_a^b \left\{ f_y[y(x)]v(x) - \left(f_z[y(x)]\right)' v(x) \right\} dx + \underbrace{f_z[y(x)]v(x)\Big|_a^b}_{\to 0}$$

$$\int_a^b \left\{ f_y[y(x)] - \left(f_z[y(x)]\right)' \right\} v(x)\,dx = 0 \qquad \forall v \in D_0$$

Fundamental lemma: If $h \in C^0[a,b]$ & $\int_a^b h(x)v(x) = 0$ $\forall v \in D_0$ then $h(x) = 0$ on $[a,b]$.

Proof: Suppose $h(x_0)>0$ for some $x_0 \in [a,b]$ (similar argument if $h(x_0)<0$). Since h is continuous $\exists [x_1, x_2] \subset [a,b]$ such that $h(x)>0$ on $[x_1,x_2]$, $x_0 \in [x_1,x_2]$, $x_1 < x_2$. Let

$$v(x) = \begin{cases} (x-x_1)^2(x-x_2)^2 & \text{for } x \in [x_1,x_2] \\ 0 & \text{elsewhere} \end{cases}$$



$v(x)>0$ on (x_1,x_2), $v \in C^1[a,b]$, $v(a)=v(b)=0 \Rightarrow v \in D_0$.

$$0 = \int_a^b h(x)v(x)\,dx = \int_{x_1}^{x_2} h(x)v(x)\,dx > 0 \quad \#$$

$\therefore h(x)=0$ on $[a,b]$ $\blacksquare$

If $(f_z[y(x)])'$ is cont. then $f_y[y(x)] - (f_z[y(x)])' = 0$

$$\longrightarrow \boxed{(f_z[y(x)])' = f_y[y(x)]}$$

Euler-Lagrange equation

$$\frac{d}{dx} f_{y'}(x,y,y') = f_y(x,y,y').$$

Lemma of du Bois-Reymond: Let $h \in C^0[a,b]$ & $\int_a^b h(x)v'(x)\,dx = 0$ $\forall v \in D_0$. Then $h(x) = c$, c some constant

Proof: Let $h_{av} = \frac{1}{b-a}\int_a^b h(t)\,dt$,

let $v(x) = \int_a^x (h(t)-h_{av})\,dt$

$\longrightarrow v(a)=0$ & $v(b) = \cdots = 0$. $v'(x) = h(x) - h_{av}$ (FTC)

$$\int_a^b h(x)v'(x)\,dx = 0 \longrightarrow \int_a^b (h(x)-h_{av})v'(x)\,dx = 0$$

$$\longrightarrow \int_a^b (h(x)-h_{av})^2\,dx = 0$$

$$\longrightarrow (h(x)-h_{av})^2 = 0 \text{ on } [a,b]$$

$$\longrightarrow h(x) = h_{av} \text{ on } [a,b]$$ $\blacksquare$

More general argument:

Let $g(x) = \int_a^x f_y[y(t)]\,dt$ then $g \in C'[a,b]$

$g'(x) = f_y[y(x)]$ (FTC)

$$0 = \int_a^b \left\{ g'(x)v(x) + f_z[y(x)]v'(x) \right\} dx$$

$$= \int_a^b \left\{ -g(x)v'(x) + f_z[y(x)]v'(x) \right\} dx + \cancelto{0}{g(x)v(x)}\Big|_a^b$$

$$= \int_a^b \left(-g(x) + f_z[y(x)] \right) v'(x)\,dx \quad \forall v \in D_0$$

By lemma du Bois-Reymond: $-g(x) + f_z[y(x)] = c$, c a const.

$f_z[y(x)] = c + g(x)$

$\rightarrow (f_z[y(x)])' = g'(x) = f_y[y(x)]$ E-L eqn.

Theorem: $y \in D$ is stationary for J if and only if $f_z[y(x)] \in C'[a,b]$ & $\underbrace{(f_z[y(x)])' = f_y[y(x)]}_{\text{Euler-Lagrange equation}}$.

Proof: Showed forward direction

Reverse direction: $(f_z[y(x)])' = f_y[y(x)]$

$$\Rightarrow \delta J(y;v) = \int_a^b \left\{ f_y[y(x)]v(x) + f_z[y(x)]v'(x) \right\} dx$$

$$= \int_a^b \left\{ \cancelto{0}{f_y[y(x)] - (f_z[y(x)])'} \right\} v'(x)\,dx + f_z[y(x)]\cancelto{0}{v(x)}\Big|_a^b = 0 \ \blacksquare$$

Example: $J(y) = \int_a^b \sqrt{1+y'(x)}\,dx$, $D = \{ y \in C'[a,b] ; y(a) = y_0, y(b) = y_1 \}$

$f = \sqrt{1+z^2}$ $\quad f_z = \frac{z}{\sqrt{1+z^2}}$, $f_y = 0$ $\quad$ E-L eqn. $\left(\frac{y'}{\sqrt{1+(y')^2}} \right)' = 0$

$\rightarrow \frac{y'}{\sqrt{1+(y')^2}} = c$, $\longrightarrow y' = m$, constant

$y = mx + b$ $\quad m, b$ constants

$y \in D \longrightarrow y$ is uniquely determined