2015 12 04(2)

PMP, Control is not bang-bang (example)

Ref: Macki & Strauss - Intro to Optimal Control

Minimize $\int_0^T (1+y_2^2)\,dt$

subject to $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases}$ $\quad \vec{y}(0) = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix},\ \vec{y}(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$|u| \le 1$ $\qquad$ $T > 0$ variable

$H = -(1+y_2^2) + p_1 y_2 + p_2 u$

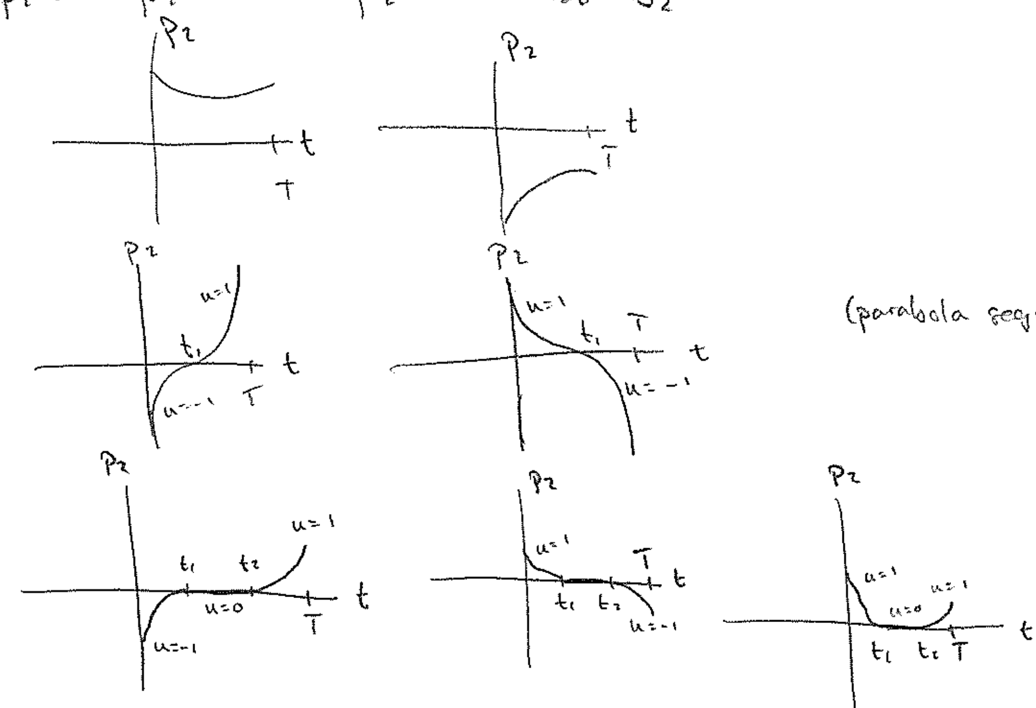
$\begin{cases} \dot{p}_1 = -H_{y_1} = 0 & \longrightarrow\ p_1 = -a \\ \dot{p}_2 = -H_{y_2} = 2y_2 - p_1 \end{cases}$

Note: $p_2, \dot{p}_2$ are continuous $\quad \ddot{p}_2 = 2\dot{y}_2 - \dot{p}_1 = 2u$

u maximizes $H \Rightarrow u = \mathrm{sgn}(p_2)$, $\quad \ddot{p}_2 = 2\,\mathrm{sgn}(p_2)$

$p_2 > 0: \ \ddot{p}_2 = 2 \longrightarrow \dot{p}_2 = 2t + a_1 \longrightarrow p_2 = t^2 + a_1 t + b_1$

$p_2 < 0: \ \ddot{p}_2 = -2 \longrightarrow p_2 = -t^2 + a_2 t + b_2$

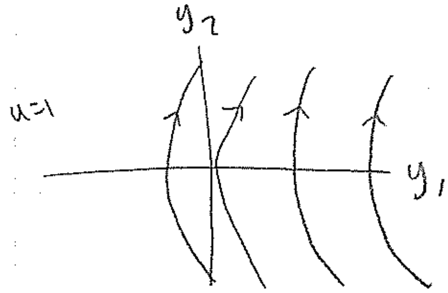


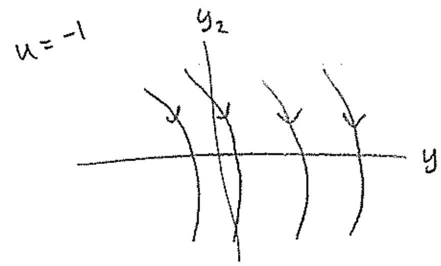
(parabola segments)

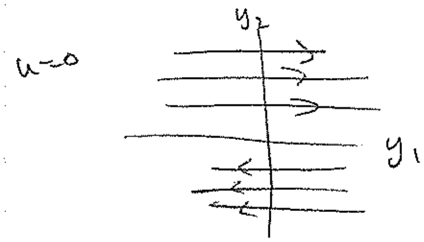
$u=1: \ \dfrac{dy_1}{dy_2} = y_2 \longrightarrow y_1 = \frac{1}{2} y_2^2 + \alpha$

$u=-1: \ \dfrac{dy_1}{dy_2} = -y_2 \longrightarrow y_1 = -\frac{1}{2} y_2^2 + \beta$

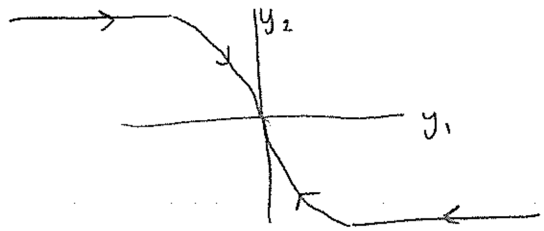
$u=0: \ y_2 = \gamma$







Must end at $\binom{0}{0}$:



If $p_2(t)=0$ on $I=[t_1,t_2]$, $a+2y_2=0$ on I

$H=-1-y_2^2-ay_2=0$ ($H=0$ on trajectories)

$-1-y_2^2+2y_2^2=0 \to y_2^2=1 \to y_2=\pm 1$ on I

Last switching time t_*:

Suppose u switches from 0 or -1 to 1 at t^*

$p_2(t_*)=0$, $p_2(t)\le 0$ on $[t_*-t, t_*]$

$\to \dot{p}_2(t_*)\ge 0$. Also $y_2(t^*)<0$, $\dot{p}_2(t)=a+2y_2(t_*)=0$

$$H|_{t=t_*}=-1-y_2^2(t_*)-ay_2(t_*)=0$$

$$-1-y_2^2(t_*)=ay_2(t_*)\le -2y_2^2(t_*)$$

$$\to y_2^2(t_*)\le 1 \leftrightarrow -1\le y_2(t_*)<0$$