Linear Systems

$\dot{\vec{y}} = A\vec{y} + B\vec{u}$ $\quad$ A, B are const.

A is $n\times n$, B is $n\times m$, $\vec{y}\in\mathbb{R}^n$, $\vec{u}\in\mathbb{R}^m$

$\hat{C}[0,T]$ are piecewise continuous functions on $[0,T]$

$\vec{u}\in(\hat{C}[0,T])^m$

Def] The system (A,B) is controllable if $\forall \vec{y}_0, \vec{y}_T$, $T>0$, $\exists \vec{u}\in(\hat{C}[0,T])^m$ st $\vec{y}(0)=\vec{y}_0$, $\vec{y}(T)=\vec{y}_T$, $\dot{\vec{y}} = A\vec{y}+B\vec{u}$.

Theorem: (A,B) is controllable if and only if

$$C = [B, AB, \ldots, A^{n-1}B]$$

has rank n.

This C is the Kalman controllability matrix.

Ex. Rocket Car

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases}, \quad \begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = \underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{A}\begin{pmatrix}y_1\\y_2\end{pmatrix} + \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{B}u \qquad \begin{array}{l} n=2 \\ AB = \begin{pmatrix}1\\0\end{pmatrix}\end{array}$$

$$C = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \text{rank } C = 2$$

$\therefore$ this system is controllable

2015 11 30

Time Optimum Problem

Minimize $\int_0^T dt = T$

subject to $\dot{\vec{y}} = A\vec{y}+B\vec{u}$, $\vec{y}(0)=\vec{y}_0$, $\vec{y}(T)=\vec{y}_T$

$\vec{u}(t)\in\mathcal{U} = \{u\in\mathbb{R}^m;\ a_k\le u_k\le b_k,\ 1\le k\le m\}$

$\forall t\in[0,1]$

$$H = -1 + \sum_{i=1}^{n} p_i\left(\sum_{j=1}^{n} a_{ij}y_j + \sum_{j=1}^{m} b_{ij}u_j\right)$$

$H=0$ on optimal solutions

2015 11 30 (2)

Optimality conditions:

$$\dot{p}_j = -H_{y_j} = -\sum_{i=1}^{n} p_i a_{ij} \qquad 1 \le j \le n$$

$$\dot{\vec{p}} = -A^* \vec{p} \qquad (A^* = A^T)$$

$$\vec{u} = \underset{\vec{v} \in \mathcal{U}}{\operatorname{argmax}}\ H(t, \vec{y}, \vec{v}, \vec{p})$$

$$= \underset{\substack{\vec{v} \in \mathcal{U} \\ a_k \le v_k \le b_k\ \forall k}}{\operatorname{argmax}} \sum_{i=1}^{n} p_i \sum_{j=1}^{m} b_{ij} v_j$$

$$u_k(t) = \begin{cases} b_k & \text{if } \sum_{i=1}^{n} p_i(t) b_{ik} > 0 \iff \vec{p}^*(t) B_k > 0 \\ a_k & \text{if } \sum_{i=1}^{n} p_i(t) b_{ik} < 0 \iff \vec{p}^*(t) B_k < 0 \end{cases}$$

Let B_k be the k^{th} column of B.

Theorem: If (A, B_k) is controllable $\forall k$ then the optimal control is bang-bang.

$\vec{p} = 0 \Rightarrow H = -1$ contradicts $H = 0$

Theorem: If the eigenvalues of A are all real & u_k is bang-bang then u_k has at most $n-1$ switching times

Ex Rocket Car $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

eigenvalues of A: $0, 0$ $\qquad \dot{\vec{y}} = A\vec{y} + Bu \quad |u| \le 1$

$u(t) = \operatorname{sgn}(at+b)$ u has at most one switching time, $n-1 = 1$

Ex Harmonic Oscillator $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 + u \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\dot{\vec{y}} = A\vec{y} + Bu \qquad |u| \le 1$

eigenvalues of A: $\pm i$

u can have arbitrarily many switching times "apparently this has a lot of applications?"

Linear Quadratic Control

Finite time

Minimize $J(\vec{u}) = \int_0^T (\vec{y}^* Q \vec{y} + \vec{u}^* R \vec{u})\,dt$

Q is symmetric positive semi-definite $(Q \geq 0)$
R is symmetric positive definite $(R > 0)$
subject to $\dot{\vec{y}} = A\vec{y} + B\vec{u}$, $\vec{y}(0) = \vec{y}_0$

$$H = -\vec{y}^* Q y - \vec{u}^* R \vec{u} + \vec{p}^*(A\vec{y} + B\vec{u})$$

$h = -H$ is convex with respect to (y,u)

$$H = -\sum_{i,j=1}^{n} q_{ij} y_i y_j - \sum_{i,j=1}^{m} r_{ij} u_i u_j + \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{n} q_{ij} y_j + \sum_{j=1}^{m} b_{ij} u_j \right)$$

$$\dot{p}_k = -H_{y_k} = \underbrace{\sum_{j=1}^{n} q_{kj} y_j + \sum_{i=1}^{n} q_{ik} y_i}_{=} - \sum_{i=1}^{n} p_i a_{ik}$$

$$\begin{cases} \dot{\vec{p}} = 2Q\vec{y} - A^*\vec{p} \\ \dot{\vec{y}} = A\vec{y} + B\vec{u} \end{cases}$$

2015 12 02

$$H_{u_k} = 0: \quad -2\sum_{j=1}^{m} r_{kj} u_k + \sum_{i=1}^{n} p_i b_{ik} = 0 \qquad 1 \leq k \leq m$$

$$-2R\vec{u} + B^*\vec{p} = 0$$

$$\left\{ \begin{array}{l} \dot{\vec{p}} = 2Q\vec{y} - A^*\vec{p} \\ \dot{\vec{y}} = A\vec{y} + B\vec{u} \\ \vec{u} = \frac{1}{2} R^{-1} B^* \vec{p} \\ \vec{y}(0) = \vec{y}_0, \ \vec{p}(T) = \vec{0} \end{array} \right\} \text{ constant coefficient system for } \vec{y}, \vec{p}$$

Theorem: Let $P(t)$ be an $n \times n$ solution to the Differential Riccati Equation (DRE)

$$\dot{P} + A^*P + PA^* - PBR^{-1}B^*P + Q = 0, \quad P(T) = O_{n\times n}.$$

Then $\vec{u}^\circ(t) = -R^{-1}B^*P(t)\vec{y}(t)$ gives the minimum cost $J(\vec{u}^\circ) = \vec{y}_0^* P(0) \vec{y}_0$, P is unique & symmetric, $P(0) \geq 0$ (pos. semi-def.).
$\vec{u}_0$ is in feedback form

Ex Minimize $\int_0^T (y^2+u^2)dt$

subject to $\dot{y}=y+u$, $y(0)=y_0$

DRE: $\dot{p}+2p-p^2+1=0$, $p(T)=0$

$A=1, B=1, Q=1, R=1$

$$\frac{dp}{dt}=p^2-2p-1$$

$$\rightarrow \int \frac{dp}{p^2-2p-1}=\int dt$$

$p^2-2p-1=(p-p_1)(p-p_2)$

$p_1=1+\sqrt{2}$, $p_2=1-\sqrt{2}$

$$\frac{1}{(p-p_1)(p-p_2)}=\frac{c_1}{p-p_1}+\frac{c_2}{p-p_2}$$

$$c_1=\frac{1}{p_1-p_2} \quad c_2=\frac{1}{p_2-p_1}$$

$p_1-p_2=2\sqrt{2}$

$$\frac{1}{2\sqrt{2}}\ln\frac{|p-p_1|}{|p-p_2|}=t+c'$$

$$\frac{p-p_1}{p-p_2}=\pm e^{2\sqrt{2}(t+c')}=Ce^{2\sqrt{2}t}$$

set $t=T$: $\frac{p_1}{p_2}=Ce^{2\sqrt{2}T} \rightarrow C=\frac{p_1}{p_2}e^{-2\sqrt{2}T}$

$$p-p_1=Ce^{2\sqrt{2}t}(p-p_2)$$

$$p(t)=\frac{p_1-p_2Ce^{2\sqrt{2}t}}{1-Ce^{2\sqrt{2}t}}$$

$u^0(t)=-p(t)y(t)$

Let $T\rightarrow\infty$: $p(t)=p_1=1+\sqrt{2}$

$u^0(t)=-(1+\sqrt{2})y(t)$

Infinite time

Minimize $J(\vec{u})=\int_0^\infty(\vec{y}^*Q\vec{y}+\vec{u}^*R\vec{u})dt$

subject to $\dot{\vec{y}}=A\vec{y}+B\vec{u}$, $\vec{y}(0)=\vec{y}_0$

Def) System (A,B) is open-loop stabilizable if $\forall \vec{y}_0\ \exists \vec{u}\in(L_2[0,\infty))^m$ such that $\vec{y}\in(L_2[0,\infty))^n$

$L_2[0,\infty)$ are functions $f(t)$ on $[0,\infty)$ such that

$$\int_0^\infty f(t)^2 dt < \infty.$$

Proposition: If (A,B) is controllable then it is open-loop stabilizable

Proof: (?) $\vec{y}_0$ $\exists\, \vec{u}_1(t)$ $0 \le t \le 1$ st $\vec{y}(1) = \vec{0}$
Let

$$\vec{u}(t) = \begin{cases} \vec{u}_1(t) & 0 \le t \le 1 \\ \vec{0} & t > 1 \end{cases}$$

Then $\vec{y}(t) = \vec{0}$, $t > 1 \rightarrow \vec{y} \in (L_2[0,\infty))^m$ □

Theorem: If (A,B) is open-loop stabilizable then $\exists$ a symm Π, $\Pi \ge 0$ giving minimum cost $J(\vec{u}^\circ) = \vec{y}_0^* \Pi \vec{y}_0$ where $\vec{u}^\circ = -R^{-1}B^*\Pi\vec{y}(t)$.
Π is the minimal solution to Algebraic Riccati Equation (ARE)

$$A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0$$

$$\vec{v}^* \Pi \vec{v} \le \vec{v}^* \Pi_1 \vec{v} \quad \forall \vec{v} \text{ for any other symm } \Pi_1 \ge 0$$

2015 12 04

Ex Minimize $J(u) = \int_0^\infty (y^2 + u^2)\,dt$

subject to $\dot{y} = y + u$, $y(0) = y_0$

$\Pi = p$, $A = 1$, $B = 1$, $Q = 1$, $R = 1$

(ARE) $2p - p^2 + 1 = 0$

$p = 1 \pm \sqrt{2}$

$p \ge 0 \Rightarrow p = 1 + \sqrt{2}$

$u^\circ(t) = -(1+\sqrt{2})y(t)$, $\quad \dot{y} = y - (1+\sqrt{2})y = \sqrt{2}y \rightarrow y = y_0 e^{-\sqrt{2}t}$

$\rightarrow \lim_{t\to\infty} y(t) = 0$ & $y \in L_2(0,\infty)$